Research Proposal

Bernoulli convolution and non-uniform self-similar measures: absolute continuity and algebraic exceptions

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1 State of the art and preliminary work

For $\beta, p \in (0,1)$ consider the infinite convolved Bernoulli measure (short: Bernoulli convolution) on the real line, given by

$$\mu_{\beta,p} = \mathop{\large\star}_{i=0}^{\infty}(p\delta_{\beta^i} + (1-p)\delta_{-\beta^i}),$$

where δ is the Dirac measure. $\mu_{\beta,p}$ is the distribution of random power series

$$X_{\beta,p} = \sum_{i=0}^{\infty} X_i \beta^i,$$

where (X_i) is a Bernoulli process given by independent identically distributed random variables taking the values in $\{1,-1\}$ with probability p resp. $1-p$.

In the last 75 years Bernoulli convolutions are intensively studied in geometric measure theory and applied in dynamical systems, number theory, harmonic analysis and fractal geometry. In (P1) You find an excellent overview of the major problems and results up to the year 2000.

It was shown by Jessen and Winter (JW), that $\mu_{\beta,p}$ is a law of pure type, either totally singular or absolutely continuous with respect to the Lebesgue measure. If $\beta < 1/2$ the measure $\mu_{\beta,p}$ is obviously supported by a Cantor set of Lebesgue measure zero and hence singular, see (KW). Moreover we know from folklore dimensional theoretical results that $\mu_{\beta,p}$ is singular if $\beta < p^p(1-p)^{1-p}$. We refer to (FA) or (PE) for an introduction to dimension theory. The main result on absolute continuity of Bernoulli convolutions is:

Theorem 1.1 *For all $p \in (0,1)$ and all $\beta \in [p^p(1-p)^{1-p}, 1)$, up to a set of Hausdorff dimension zero, the Bernoulli convolution $\mu_{\beta,p}$ is absolutely continuous.*

Let us give a brief history of this result. Erdös (E2) proved that $\mu_{\beta,0.5}$ is absolutely continuous for almost all $\beta \in (1-\epsilon, 1)$ for some $\epsilon > 0$. Solomyak (SO) obtained a major breakthrough, proving absolute continuity of $\mu_{\beta,0.5}$ for almost all $\beta \in [1/2, 1)$. A simpler proof was shortly after obtained by Peres and Solomyak (P2). Furthermore Peres and Solomyak (P3) showed that if $p \in [1/3, 2/3]$, then $\mu_{\beta,p}$ is absolutely continuous for almost all $\beta \in [p^p(1-p)^{1-p}, 1)$. In very resent preprints Hochmann (HO) and Shmerkin (S2) obtain the result presented above. In fact Hochmann proved the Bernoulli convolutions have full dimension under the assumptions and Shemerkin proved that this implies absolute continuity.
It is natural to ask if there are any executional values in the domain of absolute continuity of Bernoulli convolutions. This is in fact true:

Theorem 1.2 *If $\beta \in (0.5, 1)$ is the reciprocal of a Pisot number, the Bernoulli convolution $\mu_{\beta,p}$ is singular for all $p \in (0,1)$.*

Recall that a Pisot number is an algebraic integer with all its Galois conjugates inside the unite circle, see (BD). In the case $p = 0.5$ the theorem was proved by Erdös (E1) using the Fourier transform of the measure and the fact that powers of Pisot numbers are exponential near to integers. Lalley (LA) generalized the argument of Erdös to biased convolutions. We used this results to describe number theoretical peculiarities in the dimension theory of dynamical systems (N5). Furthermore we generalized both Theorem 1.1 and Theorem 1.2 to inhomogeneous Bernoulli convolution in collaboration with Antonios Bisbas (N1).

For $\beta_1, \beta_2, p \in (0,1)$ now consider the unique self-similar measure $\mu^p_{\beta_1,\beta_2}$ on the real line fulfilling

$$\mu^p_{\beta_1,\beta_2} = pT_1\mu^p_{\beta_1,\beta_2} + (1-p)T_2\mu^p_{\beta_1,\beta_2},$$

where $T_1x = \beta_1 x + 1$ and $T_2 x = \beta_2 x - 1$ are linear contraction on $\mathbb{R}$, see (HU). Bernoulli convolutions are uniformly self-measures, given by $\beta = \beta_1 = \beta_2$. We studied nonuniform self-similar measures in (N6) and (N2). Using the transversality techniques developed in (PS) and (P2), (P3), we were able to prove:

Theorem 1.3 *For all $p \in (0,1)$ and almost all $\beta_1, \beta_2 \in (0, 0.668)$ with*

$$\beta_1^p \beta_2^{1-p} \geq p^p(1-p)^{1-p},$$

the self-similar measure $\mu^p_{\beta_1,\beta_2}$ is absolutely continuous.

The curious bound 0.668 that appears here is due to the techniques we used. This is an approximation of the smallest double zero of powers series with coefficients in $\{-1, 0, 1\}$, see (S1). In (N4) we applied the Theorem 3.1 in theory of dynamical systems to construct absolutely continuous ergodic measures for generalized Bakers's transformations. Furthermore in (N3) we proved the existence of exceptions in the domain of generic absolute continuity of the measures $\mu^{0.5}_{\beta_1,\beta_2}$:

Theorem 1.4 *There are $\beta_1, \beta_2 \in (0, 0.668)$ with $\beta_1 \neq \beta_2$ and $\beta_1\beta_2 > 1/4$ such that the self-similar measure $\mu^{0.5}_{\beta_1,\beta_2}$ is singular.*

1.1 Project-related publications

As requested You find here six project-related publications of Jörg Neunhäuserer with links to the papers. Other cited publications can be found in the bibliography below.

(N1) On inhomogeneous Bernoulli convolution and random power series, Real Analysis Exchange, vol 36, no.1, 213-222, 2011. (with A. Bisbas). Article

(N2) A general result on absolute continuity of non-uniform self-similar measures on the real line, Fractals, vol. 16, no. 4, 299-304, 2008. Article

(N3) A construction of singular overlapping self-similar measures, Acta Mathematica Hungarica, vol. 113, 333-343, 2006. Article

(N4) Dimension theoretical properties of generalized Baker's transformations, Nonlinearity 15, 1299-1307, 2002. Article

(N5) Number theoretical peculiarities in the dimension theory of dynamical systems, Israel Journal of Mathematics 128, 267-283, 2002. Article

(N6) Properties of some overlapping self-similar and some self-affine measures, Acta Mathematica Hungarica, vol. 92, 143-161, 2001. Article

2 Objectives and work programme

2.1 Anticipated total duration of the project

The anticipated duration of the research project "Bernoulli convolution and non-uniform self-similar measures" is three years. For this time we request the support of the DFG.

2.2 Objectives

The first aim of our research project is to strength Theorem 1.1. in the following way:

Conjecture 2.1 *For all $p \in (0,1)$ and all $\beta \in [p^p(1-p)^{1-p}, 1)$ the Bernoulli convolution $\mu_{\beta,p}$ is absolutely continuous, if β is not the root of an algebraic equation with coefficients in $\{-1,0,1\}$.*

The conjecture is well known to experts in the filed, see the introduction of (HO). A proof would especially give a partita answer Question 2.5 of Peres and Solomyak (P4).

Our second objective is to find algebraic exceptions to absolute continuity of Bernoulli convolutions. Especially we conjecture:

Conjecture 2.2 *There exists roots $\beta \in (0.5, 1)$ of algebraic equations with coefficients in $\{-1,0,1\}$ that are not reciprocals of Pisot numbers such that the Bernoulli convolution $\mu_{\beta,p}$ is singular for all $p \in (0,1)$.*

Recall that a Salem number is an algebraic integer whose Galois conjugates all have modulus no greater than one, with at least one of which on the unite circle, see (BD). There appears to be a general belief, that reciprocals of Salem numbers are exceptions to Theorem 1.1, see (FE). We share this belief but conjecture in addition that there are exceptions that are neither reciprocal of Pisot numbers nor reciprocals of Salem numbers. To characterize all algebraic numbers such that Bernoulli convolution get singular seems to be an involved problem.

The third aim of the project is to remove the curious bound on absolute continuity in Theorem 1.3.

Conjecture 2.3 *For all $p \in (0,1)$ and almost all $\beta_1, \beta_2 \in (0,1)$ with*

$$\beta_1^p \beta_2^{1-p} \geq p^p(1-p)^{1-p}$$

the self-similar measure $\mu^p_{\beta_1,\beta_2}$ is absolutely continuous.

Perhaps we will be able to get an even stronger result and to estimate the Hausdorff dimension of the set of parameter for which $\mu^p_{\beta_1,\beta_2}$ is singular. At this stage of research we are not sure if there exists curves of exceptional parameters, so we do not conjecture here that the Hausdorff dimension of this set is zero.

The last challenge of the research project is to find exception in the domain of absolute continuity for non-uniform self-similar measurers:

Problem 2.1 *Describe* $\beta_1, \beta_2 \in (0,1)$ *with*

$$\beta_1^p \beta_2^{1-p} \geq p^p (1-p)^{1-p}$$

by algebraic relations, such that the self-similar measure $\mu^p_{\beta_1,\beta_2}$ *is singular for all* $p \in (0,1)$.

The proof of Theorem 1.4 does not give explicit algebraic expressions for exceptions to absolute continuity of self-similar measures. A new idea is needed to explicitly describe a class of exceptions.

The prove of each conjectures above and progress in the solution of the last problem will guarantee a publication in an outstanding mathematical journal and thus demonstrate excellence of German research in mathematics.

2.3 Work programme and proposed research methods

Our work programm consists of four phases. Each phase will approximately last nine month.

1. Phase: Proof of Conjecture 2.1

Reading the seminal work of Hochmann (HO) (and using the argument of Shmerkin (S2) in addition) we observe that Conjecture 2.1 reduces to a quite elementary question on roots of polynomials with coefficients in $\{-1,0,1\}$: Is there a constant $c>0$ such that for all different real roots α, β of polynomials with coefficients in $\{-1,0,1\}$ of degree n, we have $|\alpha-\beta| \geq c^n$? To prove that this is the case, we will first consider the literature on real algebraic integers. We think that it is perfectly possible that the separation property of real roots follows easily from known results. If not, we have a vague idea to estimate the difference of polynomials with coefficients in $\{-1,0,1\}$ analytically on nested partition to show that their roots remain separated.

2. Phase: Proof of Conjecture 2.2

Our approach to prove Conjecture 2.2 is to use the entropy of Bernoulli convolution introduced by Garsia, see (G2) and (G1). This is an upper bound on the Hausdorff dimension of the convolutions. In the case of simple Pisot numbers $\phi_n \in (1,2)$, given by the dominating root of

$$x^n - x^{n-1} - \ldots - x - 1 = 0, \quad n \geq 2,$$

there is a combinatorial strategy that provides upper bounds on the Garsia entropy, see (AZ) and (GK). This gives a new prove of the singularity of corresponding convolution

μ_β^p for $\beta = 1/\phi_n$. We will focus in our project on algebraic integers $\sigma_n \in (1, 2)$, given by the dominating root of

$$x^n - x^{n-1} - \ldots - x + 1 = 0, \quad n \geq 4,$$

and $\nu_n \in (1, 2)$ given by

$$x^n - x^{n-1} - \ldots - x^3 - 1 = 0, \quad n \geq 4.$$

Note that the numbers σ_n are Salem numbers and the numbers ν_n are neither Pisot nor Salem numbers, see (FE). For $\beta = 1/\phi_4$ and $\beta = 1/\nu_4$ we have some unpublished computational estimates on the Garsia entropy showing that the convolution μ_β^p is singular for some p with $\beta > p^p(1-p)^{1-p}$. Our algorithm is not efficient, so we do not obtain the result for $p = 0.5$, where the entropy should attain its maximum. In the proposed project we like to develop a combinatorial approach using the generating function of the Garsia entropy to prove rigorous estimates. We expect to be able to prove Conjecture 2.2 for $\beta = 1/\phi_n$ and $\beta = 1/\nu_n$ at least if n is small.

3. Phase: Proof of Conjecture 2.3

Here we will proceed in two steps. First we will try to prove the non-uniform self-similar measures have full dimension under the assumption of Conjecture 2.3. To do so, we will examine the argument of Hochmann (HO) based on an inverse theorem for entropy carefully and try to apply it in the non-uniform setting. Our strategy will be to partition the parameter space into lines and to show that the dimension of exceptions on each line is zero. This strategy has been successful to apply transversality arguments in the context of non-uniform self-similar measures. The second step is to show that full dimension implies absolute continuity of self-similar measures. The argument of Shmerkin (S2), used in the uniform setting, is based on the product structure of the Fourier transform of Bernoulli convolutions. This argument does not work for non-uniform self-similar measures and we need a completely new idea here.

4. Phase: Solution of Problem 2.4

Our prove of Theorem 1.4 above is based on a crude bound on the Hausdorff dimension of non-uniform self-similar measures with exact overlaps. To solve Problem 2.4 we have to find better estimates. Our idea is to introduce a kind of Garsia entropy in the non-uniform setting and to prove that this entropy provides an upper bound on the dimension self-similar measures with overlaps . Using algebraic relation of parameters we will estimate the entropy and obtain results on singularity for certain parameters of non-uniform self-similar measures. Perhaps as in the proof of Conjecture 2.2 combinatorial considerations will be necessary succeed.

2.4 Data handling

It is a common praxis in mathematics to publish preprints of research results online and send the articles to peer-reviewed journals for publication.

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