Parameters Estimation of a Lotka-Volterra Model in an Application for Market Graphics Processing Units

Normatov, Dzhakhongir; Mercorelli, Paolo

Parameters Estimation of a Lotka-Volterra Model in an Application for Market Graphics Processing Units

Dzhakhongir Normatov and Paolo Mercorelli
Institute of Product and Process Innovation
Leuphana University of Lüneburg
Universitaetsallee 1, 21335 Lüneburg
Germany

Abstract—In this paper, a least squares method is used to estimate parameter values in the Lotka-Volterra model. The data used are graphics processing units (GPU) shipment worldwide by three key competitors, namely Nvidia, Intel, AMD. The goal is to quantify the parameter values of a model with minimal error in order to qualitatively solve the problem and fit the raw data as closely as possible. Based on the real measurements, the predator between the competitors is recognized through the identification procedure comparing the sign of the coefficients with the original Lotka-Volterra model structure.

I. INTRODUCTION

OUR NATURAL environment is mostly in motion. In order to visualize and model the ever-changing natural environment and, more specifically, population dynamics under a variety of circumstances, the Lotka-Volterra equations have proven to be a useful tool, as long as we build our investigations on some prior assumptions. While these equations can be used for various dynamic scenarios, such as pandemics, they can also be extended to model the dynamics of two species that are sharing a natural habitat, while competing for a certain local resource, instead of one species just hunting the other. During the last years many different contributions appeared in many fields of applications and in different technical estimation and identification contexts [1], [2], [3].

The general goal of modelling is to find a simple model that fits a data set within a predetermined error bound, while still allowing specific properties to be addressed [4]. In this paper, a least squares method is used to estimate the initial parameter values in the Lotka-Volterra model. This algorithm can be used to estimate the initial value of these parameters to be used in more complex adaptive algorithms as for instance proposed in [5]. The data used is graphics processing units (GPU) shipment worldwide by three key competitors, namely Nvidia, Intel, AMD. As the parameter values of the Lotka-Volterra model describe growth, predation and competition between interacting populations, the model can be applied to competing firms in a market of GPUs or similar. The goal therefore is to quantify the parameter values of a model with minimal error in order to qualitatively solve the problem and fit the raw data as closely as possible. The paper is organized

as follows. Section II proposes a general interpretation of the Lotka-Volterra model. In Section III an application to the market of GPUs is described. Results which include a validation of the method using real data and conclusions close the paper.

II. INTERPRETATION OF THE LOTKA-VOLTERRA MODEL

The Lotka-Volterra equations, also known as the predator-prey equations, are two first-order nonlinear differential equations, often used to describe the dynamics of the biological system in which two species interact, one as a predator and the other as prey. The dynamics of the populations are described as follows:

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \alpha x(t) - \beta x(t)y(t) \\
\frac{dy(t)}{dt} &= \delta x(t)y(t) - \gamma y(t),
\end{aligned}
\tag{1}
$$

where $x(t)$ represents the number of prey (e.g. fish) and $y(t)$ represents the number of predators (e.g. bears). $\frac{dx(t)}{dt}$ and $\frac{dy(t)}{dt}$ represent the growth rates of the populations while t represents the time variable. Greek letters α, β, δ, γ are constant positive real parameters which describe the interaction of the two groups of populations. While the classic model is usually used in biology, there is a wide range of research with modified models in different areas that resemble the context of predator and prey [4], [6].

III. APPLICATION TO THE MARKET OF GPUs

Differential equations or stochastic differential equations are used for modeling random phenomena in the financial field. Due to the application of these tools in the field of financial economics, a large number of scientists research in this area, see [7], [8], [9]. It is known that the parameters of a stochastic model are always deterministically unknown and this justifies the KF approach mentioned above. The parameter estimation problem for economical models has been studied by many scientists, Yu and Phillips [10] utilized a Gaussian method to estimate the parameters of continuous time short-term interest rate models. Faff and Gray [11] considered the estimation of

short-rate models by using a method of moments. Rossi [12] took into consideration particle filters and maximum likelihood estimation for parameter estimation of Cox-Ingersoll-Ross model. In any market with competition, companies are competing for sales of their proprietary goods and services. An example of a generic Lotka-Volterra model describing the dynamics between three competing PC graphics processing units (GPUs) companies, based on a set of statistical data, is used to illustrate the practical application of the proposed method. The statistical data was retrieved from Statista [13] which includes data from 2nd quarter of 2009 to 3rd quarter of 2021, by vendor (Figure 1).
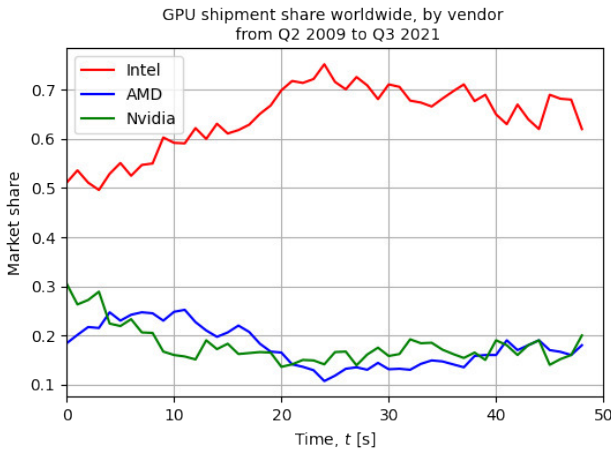

Fig. 1: GPU shipment share worldwide, by vendor, from Q2 2009 to Q3 2021. Source: [13]

From the retrieved data, it can be seen that Intel has the leading position among three firms, while other two firms have a smaller share of the market, therefore are directly competing with each other. This interaction can be modelled using the Lotka-Volterra method and the problem is formulated in terms of a nonlinear dynamical system consisting of three first order differential equations, each containing linear and quadratic terms. The system is linear with regards to the unknown parameters that must be derived from available statistical data. When applying the proposed method to the system, the unknown coefficients are estimated and the closeness of the respective solutions to the statistical data is discussed.

Given a situation where three species x, y and z compete for available resources in a system. In the market of GPUs, three main firms are competing for the attention and resources of similar customers. A Lotka-Volterra model for three competing species is generically represented by:

$$\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(a_0 + a_1 x(t) + a_2 y(t) + a_3 z(t)) \\
\frac{dy(t)}{dt} &= y(t)(b_0 + b_1 x(t) + b_2 y(t) + b_3 z(t)) \\
\frac{dz(t)}{dt} &= z(t)(c_0 + c_1 x(t) + c_2 y(t) + c_3 z(t)),
\end{aligned} \quad (2)$$

where $x(t)$, $y(t)$, $z(t)$ values are companies' shares in the market and a_i, b_i, c_i ($i = [0,3]$) are coefficients. To estimate the parameter values of the system of equations (2), the integral method described by [4] was used. As the statistical data is available, it is possible to represent the set of linear equations derived by using the integral method in the matrix form. An example of parameter value estimation for the first firm (Intel) is given below:

$$\underbrace{\begin{bmatrix} d_{1,0} \\ d_{2,1} \\ . \\ . \\ . \\ d_{n,n-1} \end{bmatrix}}_{d} = \underbrace{\begin{bmatrix} \bar{x}_{1,0} & \bar{x}^2_{1,0} & \bar{x}\bar{y}_{1,0} & \bar{x}\bar{z}_{1,0} \\ \bar{x}_{2,1} & \bar{x}^2_{2,1} & \bar{x}\bar{y}_{2,1} & \bar{x}\bar{z}_{2,1} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ \bar{x}_{n,n-1} & \bar{x}^2_{n,n-1} & \bar{x}\bar{y}_{n,n-1} & \bar{x}\bar{z}_{n,n-1} \end{bmatrix}}_{X} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\hat{a}}$$

and thus

$$\Rightarrow d = X\hat{a} \quad (3)$$

with

$$d_{j+1,j} = x(t_{j+1}) - x(t), j = 0, 1, ..., (n-1).$$

The matrix X contains the consecutive pair-wise means, namely:

$$\begin{aligned}
\bar{x}_{j+1,j} &= (x(t_{j+1}) + x(t_j))/2 \\
\bar{x}^2_{j+1,j} &= (x^2(t_{j+1}) + x^2(t_j))/2 \\
\bar{x}\bar{y}_{j+1,j} &= (x(t_{j+1})y(t_{j+1}) + x(t_j)y(t_j))/2 \\
\bar{x}\bar{z}_{j+1,j} &= (x(t_{j+1})z(t_{j+1}) + x(t_j)z(t_j))/2.
\end{aligned} \quad (4)$$

While the vector a contains the unknown parameters, a least squares method can be used to determine the vector values using the following formula:

$$\hat{a} = (X'X)^{-1}X'd \quad (5)$$

IV. RESULTS

Applying the proposed integral method, the resulting system is given as follows:

$$\begin{aligned}
\frac{dx(t)}{dt} &= x(t)(0.1113 - 0.1122x(t) + 0.1749y(t) - 0.3438z(t)) \\
\frac{dy(t)}{dt} &= y(t)(-0.1297 + 0.0041x(t) - 0.3322y(t) + 0.9385z(t)) \\
\frac{dz(t)}{dt} &= z(t)(0.0804 + 0.0496x(t) - 0.5497y(t) - 0.3117z(t)).
\end{aligned} \quad (6)$$

The estimated coefficients show that firms Intel and Nvidia are prey, while the firm AMD is the predator given its negative growth rate of -0.1297. Interestingly the share of Intel is increased by the actions of the predator firm with a coefficient of $+0.1749$ while Nvidia has a negative influence on Intel's growth rate. This can be seen by a positive coefficient of 0.0496 in the third equation. Looking at the second equation of the predator, it can be seen that the first firm is not acting as the primary prey (0.0041), whereas the third firm is the main source of the predator's share growth, as seen by a

large coefficient of 0.9385. Comparing the second and third equations (AMD vs. Nvidia) it is confirmed by the coefficient of -0.5497 that the growth of AMD comes from the decline in the third firm, Nvidia. This supports the idea that these two firms are direct competitors fighting for the remaining market share as seen on Figure 1.

In order to estimate the equilibrium point for each firm, each equation in the system of equations (6) was set equal to zero. The obvious solution for all equations is when $x(t)$, $y(t)$, $z(t)$ are equal to 0. Another solution was found at points of $x(t) = 0.63$, $y(t) = 0.11$, $z(t) = 0.17$ (see calculation in the code provided).

In order to get the values for model data, the Euler method was used. The following figures (2-4) represent the observed data, the results of modelling and forecasted data for three firms. As seen by the results, the model fits well to historical
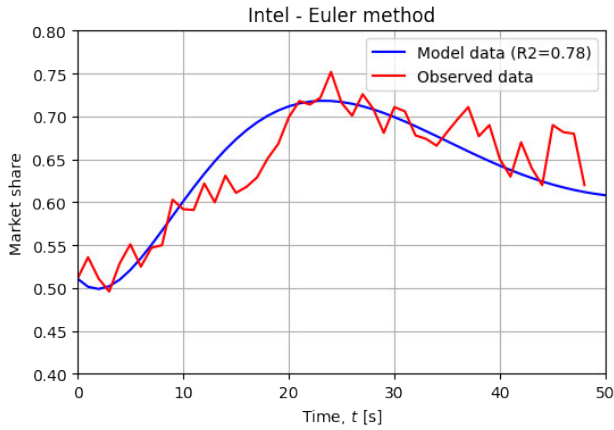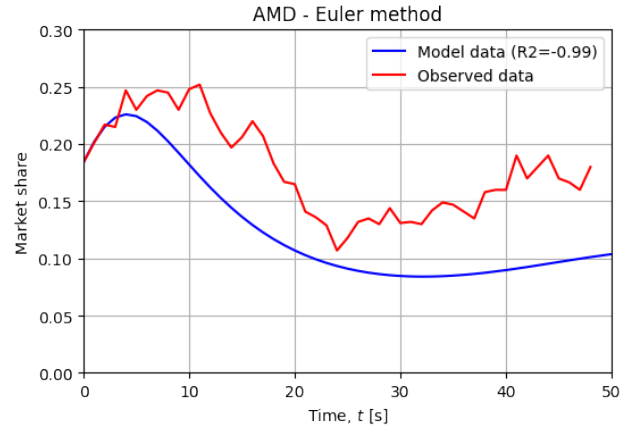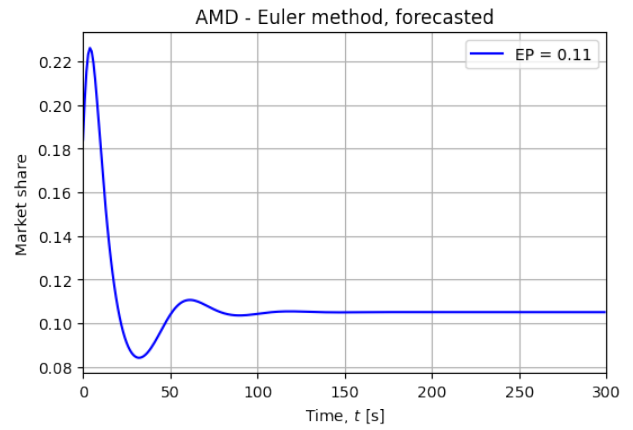

(a) Model and observed data
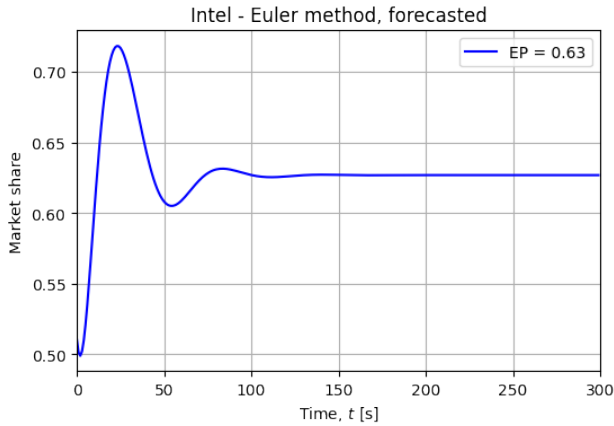
(b) Forecasted data

Fig. 2: The results for Intel

data of Intel market share, with R-squared equal to 0.78. The forecast shows that the stable state is reached at 0.63 that corresponds to the previously retrieved point of equilibrium for Intel. The forecast implies that this firm is likely to maintain its dominance in the GPU market over time.

(a) Model and observed data

(b) Forecasted data

Fig. 3: The results for AMD

As for AMD, the predator firm, it is seen that the model does not fit the observed data, which is confirmed by the R-squared value of -0.99. When finding the R-squared value, if the sum of squares of the residuals is higher than the sum of the squares of the distances of the points from a horizontal line through the mean of all Y values, then R-squared can be negative. This implies that the "best-fit" line fits worse than a horizontal line drawn through the mean of all Y values, for instance if the regression line does not follow the trend of the observed data [14]. However, as seen on Figure 3a, the modelling data nonetheless displays a similar trend as the observed data but prediction tends to be less accurate than the average value of the data set over time. The forecast implies that AMD would lose half of its market share in the near term and will remain at an equilibrium point of 0.11. However, as the model did not fit well to the observed data, it is impossible to suggest the validity of the forecast. The model for Nvidia shows a good fit with R-squared value equal to 0.78 with an equilibrium point at 0.17. The forecast shows that the firm would lose some of its market share but only by a few percentage points.
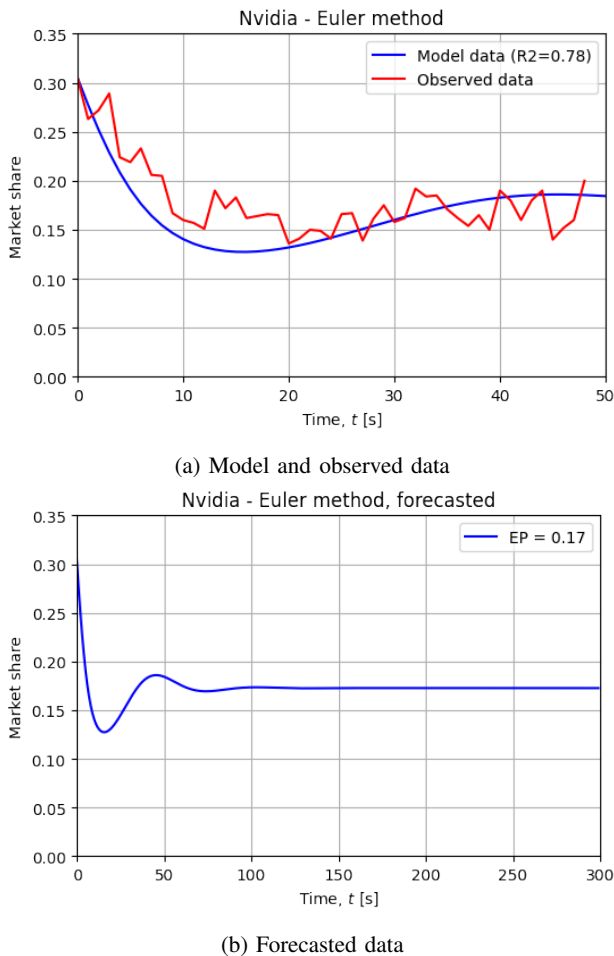
(a) Model and observed data

(b) Forecasted data

Fig. 4: The results for Nvidia

V. Conclusion

In this paper a research in the market of GPUs with three main competing firms was performed using the Lotka-Volterra model. For parameter values estimation the integral method and least squares algorithm was used. The results indicate that AMD is the predator firm while Intel and Nvidia are its prey. However, the R-squared values of the model show decent values for Intel and Nvidia, while not being a good fit for AMD, the predator firm. It is likely that more data is needed to build a better model for the predator firm. Nevertheless, the results indicate that the Lotka-Volterra model can be applied to investigate the competition of more than two firms in a free market of GPUs, but it requires further investigation and experiments.

Acknowledgments:

the scope of the Complementary Studies Programme at Leuphana University of Lüneburg during the winter semester 2021-2022. In this framework, students can explore other disciplinary and methodological approaches from the second semester onwards, focussing on additional aspects in parallel with their subjects and giving them the opportunity to sharpen skills across disciplines.

References

[1] P. Mercorelli, "A Hysteresis Hybrid Extended Kalman Filter as an Observer for Sensorless Valve Control in Camless Internal Combustion Engines," *IEEE Trans on Ind. Appl.*, vol. 48, no. 6, pp. 1940–1949, 2012. [Online]. Available: https://doi.org/10.1109/TIA.2012.2226193

[2] P. Mercorelli, "A Two-Stage Augmented Extended Kalman Filter as an Observer for Sensorless Valve Control in Camless Internal Combustion Engines," *IEEE Trans on Ind. Elects*, vol. 59, no. 11, pp. 4236–4247, 2012. [Online]. Available: https://doi.org/10.1109/TIE.2012.2192892

[3] K. Benz, C. Rech, and P. Mercorelli, "Sustainable Management of Marine Fish Stocks by Means of Sliding Mode Control," pp. 907–910, 2019. [Online]. Available: http://dx.doi.org/10.15439/2019F221

[4] P. Klopper and J. Greeff, "Lotka-Volterra model parameter estimation using experiential data," *Applied Mathematics and Computation*, vol. 224, pp. 817–825, 2013. [Online]. Available: https://doi.org/10.1016/j.amc.2013.08.093

[5] K. Benz, C. Rech, P. Mercorelli, and O. Sergiyenko, "Two Cascaded and Extended Kalman Filters Combined with Sliding Mode Control for Sustainable Management of Marine Fish Stocks," *Journal of Automation, Mobile Robotics and Intelligent Systems*, pp. 28–35, Jul. 2019. [Online]. Available: https://doi.org/10.14313/jamris/3-2020/30

[6] C. Michalakelis, T. Sphicopoulos, and D. Varoutas, "Modeling Competition in the Telecommunications Market Based on Concepts of Population Biology," *IEEE Transactions on Systems, Man, and Cybernetics, Part C (Applications and Reviews)*, vol. 41, no. 2, pp. 200–210, 2011. [Online]. Available: https://doi.org/10.1109/tsmcc.2010.2053923

[7] J. P. N. Bishwal, *Parameter estimation in stochastic differential equations*, 2008th ed., ser. Lecture Notes in Mathematics. Berlin, Germany: Springer, Oct. 2007.

[8] X. Zhang, R. D. Brooks, and M. L. King, "A Bayesian approach to bandwidth selection for multivariate kernel regression with an application to state-price density estimation," *Journal of Econometrics*, vol. 153, no. 1, pp. 21–32, Nov. 2009. [Online]. Available: https://doi.org/10.1016/j.jeconom.2009.04.004

[9] W. Xiao, W. Zhang, and W. Xu, "Parameter estimation for fractional Ornstein–Uhlenbeck processes at discrete observation," *Applied Mathematical Modelling*, vol. 35, no. 9, pp. 4196–4207, Sep. 2011. [Online]. Available: https://doi.org/10.1016/j.apm.2011.02.047

[10] J. Yu and P. C. B. Phillips, "A Gaussian approach for continuous time models of the short-term interest rate," *The Econometrics Journal*, vol. 4, no. 2, pp. 210–224, Dec. 2001. [Online]. Available: https://doi.org/10.1111/1368-423x.00063

[11] R. Faff and P. Gray, "On the estimation and comparison of short-rate models using the generalised method of moments," *Journal of Banking & Finance*, vol. 30, no. 11, pp. 3131–3146, Nov. 2006. [Online]. Available: https://doi.org/10.1016/j.jbankfin.2005.09.016

[12] G. D. Rossi, "Maximum Likelihood Estimation of the Cox–Ingersoll–Ross Model Using Particle Filters," *Computational Economics*, vol. 36, no. 1, pp. 1–16, Mar. 2010. [Online]. Available: https://doi.org/10.1007/s10614-010-9208-0

[13] Statista, "PC GPU shipment share worldwide Q2 2009 - Q3 2021, by vendor," 2021, data retrieved from Statista, https://www.statista.com/statistics/754557/worldwide-gpu-shipments-market-share-by-vendor/.

[14] D. Chicco, M. J. Warrens, and G. Jurman, "The coefficient of determination R-squared is more informative than SMAPE, MAE, MAPE, MSE and RMSE in regression analysis evaluation," vol. 7, p. e623. [Online]. Available: https://doi.org/10.7717/peerj-cs.623