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Abstract

Based on a set of seven axioms, we develop an original approach to utility under Knightian uncertainty that circumvents numerous conceptual problems of existing approaches in the literature. We understand and conceptualize Knightian uncertainty as income lotteries with known payoffs in each outcome, but unknown probabilities. This distinguishes our approach from the ambiguity approach where decision makers are assumed to have some sort of probabilistic belief about outcomes. We provide a proof that there exists a function H from the set of Knightian lotteries to the real numbers such that lottery f is preferred to lottery g if and only if $H(f) > H(g)$ and that H is unique up to cardinal transformations. We propose and illustrate one possible concrete function satisfying our axioms with a static sample decision problem and compare it to other decision rules such as maximin, maximax, the Hurwicz criterion, the minimum regret rule and the principle of insufficient reason. We find that the overall ranking of the lotteries is different from these well-known criteria, but the most preferred option is the same as with the maximin rule and a pessimistic Hurwicz individual.

JEL Classification: D81, H30

Keywords: Knightian uncertainty, deep uncertainty, decision making, envirionmental decisions, ambiguity, ambiguity aversion.

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1 Introduction

In 1921, Frank Knight coined the distinction between situations of *risk* and situations of *uncertainty* (Knight 1921). According to Knight, a situation of *risk* is one where we know the possible outcomes and their respective probabilities, whereas in a situation of *uncertainty*, we only know the outcomes but not their probabilities. While the case of risk is well studied and established, at least from a theoretical modelling standpoint, since John von Neumann's and Oskar Morgenstern's axiomatic foundation of utility under risk (von Neumann and Morgenstern 1944), the situation is fuzzier when it comes to uncertainty: The famous Ellsberg thought experiment (Ellsberg 1961) demonstrated that people tend to prefer situations of risk (i.e. known probabilities) to situations of uncertainty (unknown probabilities).[1] This behavior has been termed 'ambiguity aversion' and there are many theoretical contributions in this field which had and still continue to have a great impact on the decision theory community (Gilboa and Schmeidler 1989, Schmeidler 1989, Klibanoff et al. 2005, Maccheroni et al. 2006). Common to these approaches is their ansatz to extend the von Neumann–Morgenstern utility framework known from risk towards ambiguous situations by assuming that the decision making individual has some kind of probabilistic subjective 'belief' about the likelihood of possible outcomes. The concept has been proposed to model decision problems in climate change economics such as investing in climate change abatement when different expert groups give differing probabilistic estimates about temperature rise (e.g. Millner et al. 2010). However, there are a few problems with this approach and its concepts. First and foremost, many papers rationalize Ellsberg choices by incorporating an axiom of 'ambiguity aversion' into their framework. Yet, it is doubtable whether Ellsberg choices are a desirable feature of any theory of rational decision making. As Halevy (2007) has shown in his experimental re-examination of Ellsberg's findings, whether a test person expresses ambiguity aversion as revealed by Ellsberg choices is correlated with that person's failure to multiply probabilities. As

[1] Ellsberg's anomaly or paradox refers to the following: Assume there is an urn that contains 120 balls in total, 40 of which are blue (b) and the other 80 yellow (y) and red (r), but with unknown frequency distribution. Experiment participants are offered the following two bets: f_1 = 'win 10$ if the ball drawn from the urn is blue' vs. f_2 = 'win 10$ if the ball drawn from the urn is red' and f_3 = 'win 10$ if the ball drawn from the urn is either blue or yellow' vs. f_4 = 'win 10$ if the ball drawn from the urn is either red or yellow'. Ellsberg's finding was that the vast majority of participants preferred f_1 to f_2 and f_4 to f_3 which would imply for a probability measure underlying these choices that $P(b) > P(r)$ and $P(r) + P(y) > P(b) + P(y)$, a direct contradidiction.

Al-Najjar and Weinstein (2009) summarize one of the key findings of Halevy's study

> "Of those subjects who understood basic probability enough to reduce objective compound lotteries, 96% were indifferent to ambiguity. On the other hand, 95% of those subjects who could not multiply objective probabilities expressed ambiguity aversion."

Second, as pointed out by Al-Najjar and Weinstein (2009), the decision frameworks developed in this strand of literature that sees Ellberg choices as rational imply multiple further anomalies such as aversion to information, updating of information based on taste or sensitivity to sunk costs. Third, there is the problem of probabilistic quantification of 'beliefs' which refers to the probability distributions that the decision maker *believes* are relevant. Which probability distribution over the expected outcomes – which themselves might be hard to assess – should the decision maker include and which not? What if all the best informed persons – the experts – on a particular issue sensibly disagree? Is any probability better than no probability?

The contribution of this paper is to address these points by providing a conceptually original approach to decision making under Knightian uncertainty that does not make use of the concept of probability whatsoever. Instead, we focus on Knightian income lotteries which are distributions of monetary payoffs over different possible outcomes with completely unknown probabilities. We make a set of seven main assumptions – our base axioms – about the preference relation $\succeq$ over Knightian income lotteries, and prove that there exists a real-valued function H such that Knightian income lottery f is preferred to g if and only if $H(f) > H(g)$. This function – which we suggest to call uncertainty utility – is unique up to cardinal transformations and contains a positive real-valued parameter which we interpret as the decision maker's degree of uncertainty aversion. Conceptually, this implies that uncertainty aversion is a measure of how strong an individual dislikes spreads, i.e. unevenness, in monetary payoffs. We do not assume that individuals are uncertainty averse, but much rather, it turns out, is a very cautious attitude towards Knightian uncertainty a natural consequence of our axiomatization. Moreover, we do not relax the Sure Thing Principle, instead we show that it follows from the set of our base axioms.

The rest of this paper is organized as follows: In Section 2.1, we explain setting and notation. In Section 2.2, we state the seven base axioms and develop the main result from that, which is Proposition 1 on the existence and uniqueness of an uncertainty utility index. In Section 2.3, we propose a particular function – Rényi's generalized entropy (Rényi 1961) – as one possible functional representation of the preference relation $\succeq$ on Knightian income lotteries before we illustrate this utility index in Section 2.4 with a choice problem between three Knightian income lotteries. We also compare the result with other decision rules that have been proposed in the context of Knightian uncertainty (cf. Polasky et al. 2011): maximin, maximax, the Hurwicz criterion, Laplace's principle of insufficient reason and the principle of minimum regret. Section 3 concludes.

2 Ranking Acts under Knightian Uncertainty

In this section, we provide a brief clarification of our setting and notation before we state a set of seven base axioms that can be shown to constitute an axiomatic foundation of utility under Knightian uncertainty. We subsequently introduce a one-parameter real-valued function as one possible concrete functional representation of Knightian uncertainty preferences. In the last part of this section, we illustrate the behavior of the proposed Knightian utility index with a stylized sample decision problem between uncertain income lotteries and compare it to other decision criteria such as the maximin, maximax, minimum regret rules and the Laplace Principle.

2.1 Setting and Notation

Denote by $X \subset \mathbb{R}^n$ the set of possible states of nature and by $Y \subset \mathbb{R}^n$ the set of real-valued discrete distributions over X. By the term 'act', we mean a function $f : X \to Y$ from the set of all such acts $\mathcal{F}$. Without loss of generality, each $y \in Y$ can be thought of as a vector containing the monetary payoffs that occur in each component i of $x \in X$. Hence, we understand an act as a function that assigns a particular payoff structure y to a particular state of the world $x \in X$, the same setting that is used in Gilboa and

Schmeidler (1989)[2]. Denote by $y^f = (y_1^f, \ldots, y_n^f) \in \mathbb{R}^n$ the observed distribution resulting from a specific act f and let $\overline{y}^f = \sum_{i=1}^n y_i^f$ be the total payoff volume associated with f. Then, $s_i^f = y_i^f / \overline{y}^f$ is the payoff share of component x_i of world state x with respect to the total payoff volume $\overline{y}^f$ so that $\sum_i s_i^f = 1$ for any arbitrary f, by construction. Furthermore, denote by S^n the set containing all possible such distributions s over X, with any particular element $s \in S^n \subset \mathbb{R}^n$. If we denote by 1^n the vector $(1, \ldots, 1) \in \mathbb{R}^n$, then $\frac{1}{n}1^n$ is the uniform distribution over X. Situations of *risk* are then characterized by the possibility to assign probabilities to every entry of a specific element $s \in S^n$. If no such assignment is possible, then we have a situation of *uncertainty* (Knight 1921).

In this paper, we deal with decision problems under Knightian uncertainty. Precisely, in our setting, an act f is equivalent to a Knightian income lottery since a state of the world x is assigned a payoff structure y via f where the probabilities p_i of the payments y_i are unknown. We may thus use the words 'act' and '(Knightian) lottery' interchangeably although they are technically not quite the same. Unknown probabilities may occur in cases where the information available to the decision maker is either too vague or even conflicting to justify probabilistic 'beliefs' of any sort. Another famous argument by de Finetti (1974) is that an objectively 'true' probability distribution that any subjective belief could match or fail to match does simply not exist. In this paper, we will therefore construct a decision framework not relying on probabilities at all from a few, normatively desirable axioms and assumptions. We shall introduce these in the next section.

Before we get to the axioms of the preference relation '$\succeq$', we specify further the set of problems relevant to our framework with the help of the following two definitions.

Definition 1 (Statewise dominance). *Consider two arbitrary acts $f, g \in \mathcal{F}$ that create the payoff profiles $y^f = (y_1^f, \ldots, y_n^f)$ and $y^g = (y_1^g, \ldots, y_n^g)$. f is said to be statewise dominant over g if*

$$\exists i \in \{1 \ldots n\} : y_i^f > y_i^g \quad \wedge \quad \forall j \in \{1 \ldots n\} \setminus \{i\} : y_j^f \geq y_j^g$$

In words, act f is statewise dominant over act g if f creates a better outcome than g in at least one possible component and at least as good an outcome else. The notion of

[2]However, unlike Gilboa and Schmeidler (1989), we obviously make no assumption about measurability of acts.

statewise dominance is employed here because it is a concept that does not require any knowledge of probabilities whereas one of the key results in risk theory, the Rothschild-Stiglitz Theorem (Rothschild and Stiglitz 1970), makes use of the more common concept of *stochastic* dominance which presupposes knowledge of probabilities.

Definition 2 (Nontriviality)**.** *Denote the decision problem between two arbitrary acts f, $g \in \mathcal{F}$ by (f,g). Then (f,g) is said to be a nontrivial decision problem – not necessarily Knightian – if neither act dominates statewise.*

We assume that any decision problem considered in this paper is nontrivial in the sense of Definition 2.

The intuition behind Definition 2 is as follows: if an act f leads to a payoff profile of which any element y_i^f is at least as good as any element y_i^g and for at least one component x_i of x, it holds that $y_i^f > y_i^g$, then $f \succeq g$ trivially because, no matter what happens after choosing f, the result will be at least as good as with g. For further illustration, consider the decision between 'having 5$ for sure', formally $a : \{5\$, 5\$\}$, and $b : \{5\$, 15\$\}$. Although we do not know the probabilities associated with the states of the world that lead to the payoffs in g, it is safe to say that nobody would prefer the sure payment of 5$ to the lottery g. According to our definition, the specific decision problem (a,b) is a trivial one because no matter which state of the world actually occurs, b will always make us at least as well off as a and thus leaves no decision to scratch one's head over. Next, we introduce another important notion: degeneracy.

Definition 3 (degenerate Knightian lottery)**.** *We call Knightian lotteries degenerate if one out of the two following statements is true:*

1. *there are equal payoffs in every outcome*
2. *one outcome has a positive payoff while all other payoffs are zero.*

We have now gathered the ingredients to define the notion of uncertainty aversion in our framework.

Definition 4 (uncertainty aversion)**.** *A decision maker is said to be uncertainty averse if she – given nontriviality – always prefers a degenerate Knightian lottery with equal payoffs* *in every outcome to a non-degenerate Knightian lottery.*

Definition 4 is to say that an uncertainty averse decision maker would always be willing to pay some positive amount of money in exchange for certainty since a certain payment leaves her at least as well off as the uncertain Knightian lottery, in complete analogy to the case of risk. The cases of uncertainty neutrality and an uncertainty loving attitude can then be defined in a similar manner.

Having clarified the setting and core notions of the paper, we now have all necessary ingredients to establish our main result in the following section.

2.2 Axiomatic Foundation of Utility under Knightian Uncertainty

In this entire subsection, we make use of what is known as the Lieb-Yngvason formulation of thermodynamics (Lieb and Yngvason 1999). We shall demonstrate how their results can be used to form the axiomatic basis of a new framework for rational decision making under Knightian uncertainty. In the following, we state the axioms that we impose on the preference relation '$\succeq$' existing on the set of acts $\mathcal{F}$ where f, g, h_0 and h_1 are all acts $\in \mathcal{F}$. A 'rational' decision maker is therefore someone who agrees on these seven axioms.

Axiom 1 (Reflexivity). $f \sim f$

Axiom 2 (Transitivity). $f \succeq g$ *and* $g \succeq h$ *implies* $f \succeq h$

Axiom 3 (Consistency). $f \succeq f'$ *and* $g \succeq g'$ *implies* $f + g \succeq f' + g'$

Axiom 4 (Scaling invariance/Nonsaturation). *If* $f \succeq g$, *then* $\alpha f \succeq \alpha g \quad \forall \quad \alpha > 0$

Axiom 5 (Splitting and recombination). *For* $0 < \alpha < 1$

$$f \sim \alpha f + (1 - \alpha) f$$

Axiom 6 (Stability). *If, for some pair of acts,* f *and* $g \in \mathcal{F}$, *and for a sequence of* ε*'s tending to zero and for some arbitrary states* h_0 *and* $h_1 \in \mathcal{F}$, *it holds that*

$$f + \varepsilon h_0 \succeq g + \varepsilon h_1 \quad \Longrightarrow \quad f \succeq g$$

Axiom 7 (Completeness/weak order). *Any two acts f, $g \in \mathcal{F}$ are comparable, i.e. either $f \succeq g$ or $f \preceq g$, or both.*

For proper understanding of these assumptions on $\succeq$, we have to clarify what a mixture of lotteries or acts – as stated in Axiom 5 – means within our particular setting as it cannot mean the same as the standard compounding of lotteries which is ultimately relying on the concept of probabilities, be it objective, subjective or both. Since we do not include any probabilities in our decision framework, the compound lottery $\alpha f + (1-\alpha)g$ with $\alpha \in [0,1]$ has a different interpretation than in the case of risk. Take for example Axiom 4, the scaling invariance. It means that if the Knightian lottery {1$,2$} is at least as good as {0.50$,2.50$} to an individual, then, taking $\alpha = 50$, the very same individual should also think that {50$,100$} $\succeq$ {25$,125$}. Axiom 5 then tells us that any decision maker is assumed to be indifferent between the choice of lottery f and a combination of any two scaled versions of f. For example, assuming f = {1$,2$} and setting $\alpha = 0.4$, the decision maker would be indifferent between the lottery f and the 'lottery packet' {0.40$,0.80$} and {0.60$,1.20$} where 'packet' means she would get to play both gambles once. This indirectly reflects a very cautious attitude towards uncertainty because, in order to be indifferent between these to situations, it means that the decision maker tacitly assumes that the – unknown – odds are generally unfavorable although the actual situation might actually be the exact opposite. When choosing between lotteries as above, the very cautious decision maker who tacitly assumes unfavorable odds will reckon that she ends up having 1$ in either case and hence will be indifferent. At first glance, this seems to be in line with what Ellsberg's experiment suggested. Yet, our setting differs from Ellsberg's in that we do not assume that there is a choice between uncertainty and risk but between uncertainty and uncertainty. The consistency assumption (Axiom 3) then means that a weak preference of f to f' and g to g' implies that the lottery packet $f + g$ is also weakly preferred to the packet $f' + g'$. The stability assumption (Axiom 6) guarantees that there are no discontinuities in the preference relation which means that the presence of 'perturbatory lotteries' with small scales (in the sense of Axiom 4) tending to zero does not induce a spontaneous preference reversal. On the other hand, the axioms reflexivity, transitivity and completeness are very much standard assumptions in decision theory. Another notable point is that we do *not* assume the 'Sure Thing Principle' (Savage 1972) which means that preferences should be independent of irrelevant alternatives.

Although this is a normatively desirable feature of the uncertainty preference relation $\succeq$, we stress that we do not need it as a separate axiom because it can be shown to follow from the above axioms:

Lemma 1 (Independence of irrelevant alternatives/weak certainty independence)**.** *Let f, g, $h \in \mathcal{F}$. Then Axioms 1 through 6 imply that*

$$f + h \succeq g + h \quad \textit{implies} \quad f \succeq g$$

Proof. See Appendix A.1 □

The independence property is thus a *consequence* of Axioms 1 through 6 rather than an axiom on its own right. It reflects that alternatives that occur anyway should not matter for decision making. This is a very important feature that seems to pose a problem to the ambiguity aversion literature (Al-Najjar and Weinstein 2009).

On a technical remark, we could at first *assume* completeness instead of presupposing it in a separate axiom as we have done here. But in that case, as Lieb and Yngvason (1999) have pointed out, we would need a total of 15 axioms to guarantee the existence of a unique 'uncertainty utility index' representing the uncertainty preferences of any rational (in the sense of Axioms 1 through 7) decision maker that we state in the following.

Proposition 1 (Existence and uniqueness of an uncertainty utility index)**.** *Let $\succeq$ be a binary relation on $\mathcal{F}$. Then the following statements are equivalent:*

1. *$\succeq$ satisfies Axioms 1 – 7.*
2. *$\succeq$ represents entropic preferences.*
3. *There exists a function $H : S^n \to \mathbb{R}$ such that*

$$H(s^f) \geq H(s^g) \quad \Longleftrightarrow \quad f \succeq g$$

H is unique up to cardinal transformations $H'(s) = aH(s) + b$ where $a, b \in \mathbb{R}$ and $a > 0$.

Proof. Proof by analogy to Lieb and Yngvason (1999), see Appendix A.2 □

Hence, the imposition of Axioms 1 through 7 on the preference relation $\succeq$ on $\mathcal{F}$ implies the existence of a unique function from the set of Knightian lotteries S^n to the real numbers that reflects the preferences over Knightian lotteries (uncertainty preferences) by assigning the greatest real number to the Knightian lottery that the DM values most. We shall call this function $H(s)$ the uncertainty utility index of an individual DM with respect to a situation of Knightian uncertainty described by the payoff structure s. From Proposition 1, it follows that this function is of the von Neumann-Morgenstern type of utility functions. That is, being unique up to cardinal transformations, differences in utilities are meaningful for each individual, but not interpersonally comparable (cf. Roemer 1996). Moreover, the axioms that we have stated here to define rational decision making under Knightian uncertainty hold true particularly for any entropy function. Therefore, we may call any preference relation satisfying Axioms 1 through 7 an *entropic* preference relation. That being said, we have the following observation:

Proposition 2. *Each entropic preference is uncertainty averse.*

This is to say that aversion to Knightian uncertainty manifests itself in an aversion to unevenness in payoff structures. Intuitively, a more even payoff structure is equivalent to more certainty up to the point of absolute certainty when the payoff distribution over the possible states of the world is perfectly even. A numerical equivalent for a distribution's degree of (un-)evenness is exactly what an information entropy function yields. Therefore, we will investigate in the following subsection one possible such measure that contains a real-valued positive parameter capturing the decision maker's degree of uncertainty aversion.

2.3 Rényi's Generalized Entropy as a Numerical Equivalent of Uncertainty Averse Preferences

In this section, we propose a generalized one-parameter entropy measure known from information theory to represent the decision maker's uncertainty preferences. The measure contains a positive, real-valued parameter which reflects the individual's degree of uncertainty aversion. We start by technically introducing the function – Rényi's generalized entropy (Rényi 1961) – before we elaborate on its interpretation in the context of deciding between Knightian lotteries.

Definition 5 (Rényi's generalized entropy)**.** *For $n \in \mathbb{N}$, $s \in S^n$ and $\alpha > 0$, the following function $H^n_\alpha : S^n \to \mathbb{R}$ is called entropy of order α, or equivalently Rényi's generalized entropy:*

$$H^n_\alpha(s) = \begin{cases} \frac{1}{1-\alpha} \ln \left(\sum_{i=1}^n s_i^\alpha\right) & : \alpha > 0, \quad \alpha \neq 1 \\ -\sum_{i=1}^n s_i \ln s_i & : \alpha = 1 \end{cases} \tag{1}$$

Obviously, the case $\alpha = 1$ is a special case of the general expression in the upper row of the definition. Since $\lim_{\alpha \to 1} H^n_\alpha(s)$ is of the structure that allows to employ l'Hôpital's rule, the expression for $H^n_1(s)$ which is also known as Shannon's entropy follows from that.

Proposition 3 (Properties of Rényi's generalized entropy)**.** *Rényi's generalized entropy (Equation 1) has the following properties for every $\alpha > 0$, $n,m \in \mathbb{N}$, $s \in S^n$ and $r \in S^m$:*

1. *Continuity: $H^n_\alpha(s)$ is a continuous function of s.*
2. *Symmetry: $H^n_\alpha(s) = H^n_\alpha(Ps)$ for every permutation matrix P.*
3. *Maximality: $H^n(\frac{1}{n}1^n) > H^n(s)$ for every $s \in S^n \setminus \left\{\frac{1}{n}1^n\right\}$.*
4. *Minimality: $H^n_\alpha(s) = 0$ if $s = (1,0,\ldots,0)$.*
5. *Additivity: $H^{mn}_\alpha(r \circ s) = H^m_\alpha(r) + H^n_\alpha(s)$ where $r \circ s$ is the product distribution consisting of shares $r_i \circ s_j$, $i = 1 \ldots m$, $j = 1 \ldots n$.*

Proof. See Appendix A.3 □

The symmetry property states that the sequence of the payoff shares that result from an act does not affect the value of H^n_α so that it does not matter in what sequence these shares are numbered. The maximality property tells us that H^n_α reaches its unique maximum for a completely uniform distribution. From our proof, it follows that this maximum value is equal to $\ln n$ and hence independent of α. Conversely, the minimality property states that H^n_α becomes minimal for a situation where one substate of the world has maximum market share and this minimal value is zero. Additivity states it does not change the value of H^n_α if one divides the payoff structure into subparts and calculates the entropy as sum of these subparts.

Proposition 4 (Influence of the degree of uncertainty aversion α on the utility index $H^n_\alpha(s)$)**.** *For any arbitrary non-degenerate Knightian lottery s, a higher degree of uncertainty aversion implies ceteris paribus a lower level of uncertainty utility. Conversely, a lower degree of uncertainty aversion implies ceteris paribus a higher level of uncertainty utility. Formally,*

$$\frac{\partial H^n_\alpha(s)}{\partial \alpha} < 0 \tag{2}$$

for any non-degenerate s.

Proof. See Appendix A.4. □

Proposition 4 implies that the degree of aversion to Knightian uncertainty α determines how much the distribution's evenness is weighted. In general, the act creating the more uneven payoff distribution will ceteris paribus provide the smaller utility H^n_α to the DM. And the evaluation of an act in terms of utility depends on α. Hence, we have as a general rule for any distribution $s \in S^n \setminus \left\{\frac{1}{n}1^n\right\}$

$$H^n_{\alpha_1}(s) > H^n_{\alpha_2}(s) > H^n_{\alpha_3}(s) > \ldots > H^n_{\alpha_\infty}(s) \quad \text{where} \quad \alpha_1 < \alpha_2 < \alpha_3 < \ldots < \alpha_\infty$$

We illustrate this behavior in Figure 1.

In general, two acts to be compared need not live in the same dimension. For example, it might happen that $f : \mathbb{R}^n \to \mathbb{R}^n$ while $g : \mathbb{R}^m \to \mathbb{R}^m$ and $n \neq m$. We know from the maximality property of Rényi's generalized entropy that the maximum possible index value is $\hat{H}^n = H^n(\frac{1}{n}1^n) = \ln n$. One way of establishing general comparability is thus a normalization.

Definition 6 (Normalized Rényi's generalized entropy)**.** *For any act $f : \mathbb{R}^n \to \mathbb{R}^n$, we define*

$$\widetilde{H}_\alpha(s^f) := \frac{H^n_\alpha(s^f)}{\ln n} \tag{3}$$

so that $\widetilde{H}_\alpha \in [0,1]$ *and*

$$\widetilde{H}_\alpha(s^f) = 1 \quad \Longleftrightarrow \quad s^f = \frac{1}{n}1^n.$$

Thus, for the hypothetical situation of certainty, we have $\widetilde{H}_\alpha(\frac{1}{n}1^n) = 1$. As long as the decision problem fulfills Definition 2, certainty is thus the state most desirable to the uncertainty averse individual. On the other hand, maximum uncertainty in the sense that one particular state of the world gets a payoff share of one is the least desirable state, and, by the Minimum principle from Proposition 3, $H^n_\alpha(s) = \widetilde{H}^n_\alpha(s) = 0$. And because of Proposition 1, we know that it is then possible to assign specific utility numbers to any arbitrary payoff distribution s which reflect its desirability to the individual.

In Figure 1, we plot the utility index for the case of just two possible outcomes which means

$$H^2_\alpha(s) = \frac{1}{1-\alpha}\ln\left[s^\alpha + (1-s)^\alpha\right]; \quad \alpha > 0 \neq 1 \tag{4}$$

since knowing one share $s_1 = s$ determines the other share $s_2 = 1 - s$. Apart from the behavior that is clear already from Proposition 3 that we have just discussed, Figure 1 illustrates that the numerical utility that a DM attaches to a Knightian lottery critically depends on her attitude towards uncertainty α. For small α between 0 and 1, the DM attaches a relatively high utility even to very small shares. Greater values of s imply higher levels of utility but at an ever smaller positive marginal utility. While being uncertainty averse – the best situation in terms of utility is still the situation of certainty at $s = 0.5$ – the utility gain from an additional marginal unit of s is very small for greater s closer to $s = 0.5$ compared to smaller s closer to $s = 0$. We have $H'_\alpha(s) > 0$ and $H''_\alpha(s) < 0$. On the other hand, for $\alpha > 1$, we observe that $H'_\alpha(s) > 0$ and $H''_\alpha(s) > 0$ for $s < s^*$ and $H'_\alpha(s) > 0$ and $H''_\alpha(s) < 0$ for $s > s^*$. That is, we observe that the curvature of the curve changes at some point s^* depending on α and that the locus of s^* moves towards $s = 0.5$ with increasing α. Hence, the more uncertainty averse the individual, the higher the level of s up to which she enjoys an over proportionally high marginal utility from an additional marginal unit of s. Moreover, for the case $\alpha > 1$, the DM attaches very low utility values to small shares s with small positive marginal utility compared to the case of low degrees of uncertainty aversion $0 < \alpha < 1$.

Clearly, an individual's current wealth level should matter in a decision framework that deals with Knightian uncertainty. After all, it does make a difference in terms of our well-being whether we face a potential loss of, say, 20000 \$ from a current wealth level of 100000 \$ or from a mere 25000 \$. This is incorporated in our formalization. Consider the numbers that we have just stated from the viewpoint of outcomes and shares: in the first situation, we face the Knightian lottery $f = \{100000\$, 80000\$\}$, whereas in the second case, we look at $g = \{25000\$, 5000\$\}$, so that we have $s^f = (\frac{5}{9}, \frac{4}{9})$ and $s^g = (\frac{5}{6}, \frac{1}{6})$ in terms of shares. If the individual accepts our axioms about preferences over Knightian lotteries and if we assume her numerical preference is of the form given in Equation 4 with $\alpha = 5$, it means that in the wealthier situation, her utility level is $H_5^2(s^f) = 0.664$ ($\widetilde{H}_5^2(s^f) = 0.958$) compared to $H_5^2(s^g) = 0.228$ ($\widetilde{H}_5^2(s^f) = 0.329$) in the situation with much less initial wealth resulting in the Knightian lottery g.

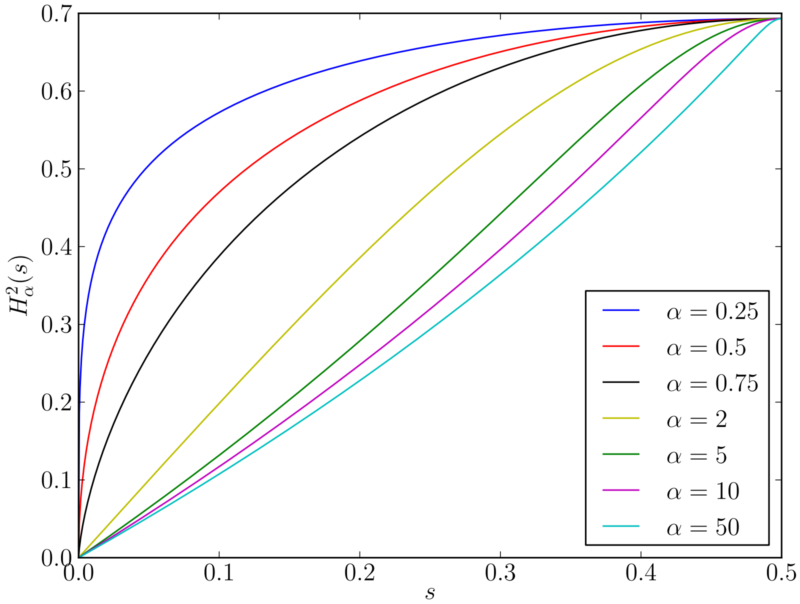


Figure 1: Numerical representation of uncertainty averse preferences as given by Rényi's generalized entropy for various degrees of uncertainty aversion and only two possible outcomes such that $s_1 = s$ and $s_2 = 1 - s$. In such a world where $n = 2$, the maximum possible H value is $H_\alpha^n(0.5) = \ln 2 \approx 0.69$.

2.4 Illustration: Comparison of Decision Rules

This section illustrates our proposed preference ordering index decision problems within the formal scope of Definition 2 with a concrete example. We compare our decision criterion to other decision criteria proposed in the literature to deal with Knightian uncertainty. These criteria include the maximin rule and its optimistic counterpart, the maximax rule, Laplace's principle of insufficient reason, the rule of minimum regret and the Hurwicz criterion which is a linear combination of maximin and maximax (Polasky et al. 2011) which weights possible maximum and minimum payoffs in each state according to the decision maker's optimism.

While terms such as maximin and maximax are self-explanatory, this is less true for the other three decision rules mentioned. Pierre-Simon Laplace's principle of insufficient reason states that there is no reason to assume that one specific state of the world is more probable than another one when probabilities are unknown (Laplace 1820). Hence, they should all be given equal weight so that Laplace's principle amounts to choosing the alternative that generates the highest average payoff. The rule of minimum regret is based on the idea to minimize the maximally possible 'regret': for each possible state of the world, the act that leads to the highest payoff is set as reference point relative to which the 'regret' is calculated as possible payoff foregone given that the respective state of the world is realized. The alternative that minimizes the maximum possible regret is considered the best choice in this decision framework. Eventually, the Hurwicz rule generalizes the maximin and maximax criteria: for each alternative $k \in \mathcal{F}$, the score

$$\Phi(y^k) = \lambda \max_i(y_i^k) + (1-\lambda) \min_i(y_i^k)$$

is calculated and compared to the alternatives's scores. The associated decision rule is $\max_k \Phi(y^k)$. λ thus reflects the individual's optimism as a greater λ gives more weight to the maximum payoff of the lottery and hence less to the minimum. In consequence, $\lambda = 1$ corresponds to the maximax rule while $\lambda = 0$ leads to the maximin criterion.

As exemplary decision problem, we borrow the following example, slightly altered, from Dörsam (2003): an individual has to take a decision between three acts – f, g and h – where $x \in \mathbb{R}^4$. The acts are known to generate the following payoff profiles (in monetary units)

$$\begin{aligned} f &: \{60, 30, 50, 60\} \\ g &: \{10, 10, 10, 140\} \\ h &: \{5, 100, 120, 130\} \end{aligned}$$

It is easy to verify that these Knightian income lotteries fulfill Definitions 1 and 2. f is very even but does neither have an especially large maximum payoff nor a particularly low minimum possible payoff. In fact, it guarantees the maximal minimum payoff out of the three alternatives. Hence, the maximin criterion would select lottery f. Lottery g offers the potentially highest possible win out of all three uncertain prospects but it only does so in one out of four possible states of the world whereas in the other three states, we end up having only 10 monetary units. Obviously, the maximax criterion would rate g highest and f lowest. Lottery h features the smallest minimum but otherwise it offers three potentially large payoffs as compared to f and g. Hence, Laplace's principle of insufficient reason ranks h highest, likewise does the rule of minimum regret. The advice that the Hurwicz criterion gives us critically depends on the choice of λ. A rather pessimistic individual ($\lambda = 0.1$) would choose lottery f while for any $\lambda \geq 0.2$, lottery g would be preferred. We give the complete rankings of acts in Table 1.

Table 1: Orderings over the Knightian lotteries f, g and h that result from different decision rules.

decision criterion	choice ordering
maximin	$f \succ g \succ h$
maximax	$g \succ h \succ f$
Laplace principle	$h \succ f \succ g$
minimum regret	$h \succ f \succ g$
Hurwicz	$\lambda = 0.1$: $f \succ g \succ h$
	$\lambda = 0.8$: $g \succ h \succ f$

In Table 2, we illustrate how the H_α^4 scores of the three Knightian lotteries f, g and h change as the parameter of uncertainty aversion α is increased. In our framework, a comparison of differences in utility is meaningful for the same individual due to Proposition 1 (uniqueness up to cardinal transformations). We see that, although the preference over the lotteries always remains $f \succ h \succ g$, the impact of bad outcomes drastically changes with α: at a very low level of uncertainty aversion ($\alpha = 0.1$), the normed utilities $\widetilde{H}$ provided by the three uncertain prospects are within a range of 0.056 from lottery f (best) to lottery g (worst), whereas at high levels of uncertainty aversion, the range is 0.773. Thus, an individual relatively uncaring towards Knightian uncertainty would gain relatively little in terms of utility when swapping from prospect g to prospect f. On the other hand, the very same swap would mean an over six times higher level of utility to a highly uncertainty averse decision maker.

Table 2: H^4_α scores of the three Knightian lotteries f, g and h for different degrees of uncertainty aversion α. Also given are the normalized scores as defined in Definition 6. The resulting preference ordering is $f \succ h \succ g$.

uncertainty aversion	uncertainty utility					
α	$H^4_\alpha(f)$	$\widetilde{H}_\alpha(f)$	$H^4_\alpha(g)$	$\widetilde{H}_\alpha(g)$	$H^4_\alpha(h)$	$\widetilde{H}_\alpha(h)$
0.1	1.383	0.998	1.306	0.942	1.338	0.965
0.5	1.369	0.986	0.983	0.709	1.215	0.876
1	1.354	0.977	0.660	0.476	1.151	0.830
3	1.309	0.944	0.291	0.210	1.103	0.796
5	1.283	0.925	0.243	0.175	1.090	0.786
10	1.252	0.903	0.216	0.156	1.070	0.772
20	1.230	0.887	0.204	0.147	1.048	0.756
50	1.214	0.876	0.198	0.143	1.025	0.739

3 Conclusion

In this paper, we have provided a new representation of preferences over Knightian income lotteries, which we have defined as income lotteries with unknown probabilities. Specifically, we have given an axiomatic foundation of a utility index that maps Knightian lotteries to the positive reals resting on seven base axioms. Two features of our axiomatization are especially desirable: (1) We do not relax the Sure Thing Principle which cures the problem of sensitivity to irrelevant sunk costs that many approaches from the ambiguity aversion strand of literature have and which was pointed out by Al-Najjar and Weinstein (2009); (2) we do not assume uncertainty aversion from the beginning, instead a very cautious attitude towards spreads in Knightian income lotteries follows as a natural consequence of our axiomatization. We have demonstrated that this utility index is unique up to cardinal transformations, paralleling the seminal approach to utility under risk by von Neumann and Morgenstern (1944). We have suggested to use a function known from physics – Rényi's generalized entropy Rényi (1961) – as a possible functional representation of preferences over Knightian lotteries. This representation has the appeal of containing a positive, real-valued parameter that captures the individual's dislike of spreads in Knightian income lotteries. We have suggested an interpretation of this parameter as the individual's degree of uncertainty aversion which is conceptually original and quite different from the concepts of 'ambiguity aversion' in the literature which predominantly rely on probabilistic conceptualizations of prior 'beliefs'. With these new concepts of uncertainty aversion and utility under Knightian uncertainty, we circumvent several conceptual issues in existing modelling approaches such as aversion to information (cf. Al-Najjar and Weinstein 2009), introduction of probability as one more parameter with possibly large quantification uncertainties and the sensitivity to irrelevant sunk costs mentioned before. We have illustrated our uncertainty utility index with a set of three sample Knightian income lotteries and compared the resulting rankings to five well-known approaches to decision under Knightian uncertainty which are the maximin and maximax rules, the Hurwicz criterion, Laplace's principle of insufficient reason and the rule of minimum regret. From this illustration, we learn that our uncertainty utility index produces a ranking of the sample Knightian income lotteries different from the other five decision rules. However, the most preferred lottery coincides with the one preferred by an individual with maximin preferences and with the choice a very pessimistic Hurwicz individual would make.

A Appendix: Proofs

This Appendix provides the detailed proofs of the propositions and lemmata that we have omitted in the main body of the text for better readability.

A.1 Proof of Lemma 1

Assume without loss of generality that $f \succeq g$. Let furthermore $\epsilon = \frac{1}{n}$ where $n \in \mathbb{N}$. Then, we have by Axioms 1 and 5

$$f + \epsilon h \sim (1 - \epsilon)f + \epsilon f + \epsilon h$$

By Axiom 4, if $f \succeq g$, then also $\epsilon f \succeq \epsilon g$. Moreover, we know that

$$f - \epsilon f + \epsilon h \sim (1-\epsilon)f + \epsilon h$$

Combining the last equation with the assumed relation of acts f and g, we have by Axiom 3

$$\begin{aligned} f + \epsilon h &\succeq (1-\epsilon)f + \epsilon g + \epsilon h \\ &\sim (1-2\epsilon)f + \epsilon f + \epsilon g + \epsilon h \end{aligned}$$

From here, by the same argument as above, we arrive at

$$f + \epsilon h \succeq (1-2\epsilon)f + 2\epsilon g + \epsilon h$$

Repeating this n times yields

$$f + \epsilon h \succeq g + \epsilon h$$

And from here, we may conclude by Axiom 6 that $f \succeq g$ indeed.

□

A.2 Proof of Proposition 1

The original proof is due to Lieb and Yngvason (1999). Here, we will transfer it to our framework of preferences over Knightian lotteries and establish the result in our context.

The proof will be carried out in four steps, each of which will be formulated in a separate lemma. First, we show in Lemma 2 that for any $f \in \mathcal{F}$ there exists a number $H_{\mathcal{F}}(f)$ on the space of Knightian lotteries $\mathcal{F}$ that is well-defined and bounded above. In Lemma 3, we demonstrate the equivalence of $\preceq$ on $\mathcal{F}$ and $\leq$ on $\mathbb{R}$. Third, we show in Lemma 4 that $H_{\mathcal{F}}(f)$ is unique and in the last step, Lemma 5 establishes that this uniqueness holds up to cardinal transformations.

Lemma 2. *Suppose that* f_0 *and* $f_1 \in \mathcal{F}$ *with* $f_0 \prec f_1$ *and define for* $\lambda \in \mathbb{R}$

$$H_\lambda = \{f \in \mathcal{F} : (1-\lambda)f_0 + \lambda f_1 \preceq f\}$$

Then

1. $\forall f \in \mathcal{F}$*, there is a* $\lambda \in \mathbb{R}$ *such that* $f \in S_\lambda$
2. $\forall f \in \mathcal{F}$, $\sup\{\lambda : f \in S_\lambda\} < \infty$

In words, (1) for every Knightian lottery, or equivalently act f, there exists a real number such that $f \in H_\lambda$ and (2) this real number is bounded above.

Proof. 1. If $f_0 \preceq f \Rightarrow f \in S_0$ by Axiom 2. For general f, we claim for some $\alpha \geq 0$

$$(1+\alpha)f_0 \preceq \alpha f_1 + f \tag{5}$$

and hence

$$(1-\lambda)f_0 + \lambda f_1 \preceq f \quad \text{with} \quad \alpha = -\lambda$$

If Equation 5 were not true, then $\alpha f_1 + f \preceq (1+\alpha)f_0 \quad \forall \alpha > 0$ and so, by Axioms 4 and 5

$$f_1 + \frac{1}{\alpha} f \preceq f_0 + \frac{1}{\alpha} f_0$$

By Axiom 6, this would imply $f_0 \preceq f_1$, in contradiction to the assumption

2. This is essentially the same argument, i.e. proof by contradiction: If $\sup\{\lambda : f \in S_\lambda\} = \infty$, then we would have for some sequence of λ's tending to ∞

$$(1-\lambda)f_0 + \lambda f_1 \preceq f$$

which would imply by Axioms 3 and 5 that

$$f_0 + \lambda f_1 \preceq f + \lambda f_0$$

and by Axiom 4

$$\frac{1}{\lambda} f + f_1 \preceq \frac{1}{\lambda} f + f_0$$

which would imply by the stability axiom that $f_1 \preceq f_0$.

$\square$

In the next step, we assume Knightian lotteries $f_0, f_1 \in \mathcal{F}$ with $f_0 \preceq f_1$ and define for arbitrary $f \in \mathcal{F}$

$$H_{\mathcal{F}}(f) := \sup \{\lambda : (1-\lambda) f_0 + \lambda f_1 \preceq f\} \tag{6}$$

the canonical utility on $\mathcal{F}$ with reference points f_0 and f_1 in the space of lotteries $\mathcal{F}$. Then Lemma 2 guarantees that $H_{\mathcal{F}}(f)$ is well-defined and bounded above.

Lemma 3 (Equivalence of $\preceq$ and $\leq$)**.** *Assume $f_0 \preceq f_1$ as before and $a_0, a_1, a_0', a_1' \in \mathbb{R}$ with $a_0 + a_1 = a_0' + a_1'$. Then the following are equivalent*

1. $a_0 f_0 + a_1 f_1 \preceq a_0' f_0 + a_1' f_1$
2. $a_1 \leq a_1'$ *(and hence $a_0 \geq a_0'$)*

Furthermore, $\sim$ holds in 1. if and only if $a_1 = a_1'$ and $a_0 = a_0'$.

Proof. Assume that $a_0 + a_1 = a_0' + a_1' = 1$ and that all a's are strictly positive.
1. $\Rightarrow$ 2.: We write $\lambda = a_1$ and $\lambda' = a_1'$. We deliberately assume that $\lambda > \lambda'$ which violates 2. above to show that this assumption leads to a contradiction. If indeed $\lambda > \lambda'$, then we have

$$(1-\lambda) f_0 + \lambda f_1 \preceq (1-\lambda') f_0 + \lambda' f_1$$

and by Axioms 3 and 5, we get

$$(1-\lambda) f_0 + \lambda' f_1 + (\lambda - \lambda') f_1 \preceq (1-\lambda) f_0 + (\lambda - \lambda') f_0 + \lambda' f_1$$

From this, applying Axioms 3 and 5 again, we arrive at $(\lambda - \lambda') f_1 \preceq (\lambda - \lambda') f_0$ which yields by Axiom 4 that $f_1 \preceq f_0$ which is the contradiction we were looking for.
2. $\Rightarrow$ 1.:

$$\begin{aligned} (1-\lambda) f_0 + \lambda f_1 &\overset{\text{A3/A5}}{\sim} (1-\lambda') f_0 + (\lambda' - \lambda) f_0 \lambda f_1 \\ &\overset{\text{A3/A4}}{\preceq} (1-\lambda') f_0 + (\lambda' - \lambda) f_1 + \lambda f_1 \\ &\overset{\text{A3/A5}}{\sim} (1-\lambda') f_0 + \lambda' f_1 \end{aligned} \tag{7}$$

which only holds for $\lambda' > \lambda$. $\square$

Lemma 4. *(Uniqueness of canonical utility $H_{\mathcal{F}}$) Let $H_{\mathcal{F}}$ denote the canonical utility from Equation 6 on $\mathcal{F}$ with respect to the reference lotteries $f_0 \preceq f_1$. If $f \in \mathcal{F}$, then*

$$\lambda = H_{\mathcal{F}}(f)$$

is equivalent to

$$f \sim (1-\lambda)f_0 + \lambda f_1$$

Proof. First, if $\lambda = H_{\mathcal{F}}(f)$, then by definition of the supremum, there is a sequence $\varepsilon_1 \geq \varepsilon_2 \geq \ldots \geq 0$ converging to zero such that

$$(1-(\lambda-\varepsilon_n))f_0 + (\lambda-\varepsilon_n)f_1 \preceq f \quad \forall n$$

By Axiom 5

$$\begin{aligned}(1-\lambda)f_0 + \lambda f_1 + \varepsilon_n f_0 \;&\sim\; (1-\lambda+\varepsilon_n)f_0 + (\lambda-\varepsilon_n)f_1 + \varepsilon_n \\ &\preceq\; f + \varepsilon_n f_1 \end{aligned} \tag{8}$$

By Axiom 6, we get

$$(1-\lambda)f_0 + \lambda f_1 \preceq f$$

On the other hand, since λ is the supremum we have by Axiom 7

$$f \preceq (1-(\lambda+\varepsilon))f_0 + (\lambda+\varepsilon)f_1 \quad \varepsilon > 0$$

which means

$$f + \varepsilon f_0 \preceq (1-\lambda)f_0 + \lambda f_1 + \varepsilon f_1$$

and so, by Axiom 6 again

$$f \preceq (1-\lambda)f_0 + \lambda f_1 \quad \Rightarrow f \sim (1-\lambda)f_0 + \lambda f_1 \quad \text{when} \quad \lambda = H_{\mathcal{F}}(f)$$

If, conversely, $\lambda' \in [0{,}1]$ is such that

$$f \sim (1-\lambda')f_0 + \lambda' f_1$$

then by Axiom 2

$$(1-\lambda)f_0 + \lambda f_1 \sim (1-\lambda')f_0 + \lambda' f_1$$

and thus $\lambda = \lambda'$ by Lemma 3. □

Hence, for every $f \in \mathcal{F}$, there is a unique $\lambda \in \mathbb{R}$, namely $\lambda = H_{\mathcal{F}}(f)$, such that

$$f \sim (1-\lambda)f + \lambda f \sim (1-\lambda)f_0 + \lambda f_1 \tag{9}$$

Put differently, any Knightian lottery f is always representable by a linear mixture of two arbitrary, non-identical lotteries from $\mathcal{F}$ with mixture parameter λ.

In the following last Lemma, we demonstrate that $H_{\mathcal{F}}$ is unique up to cardinal transformations.

Lemma 5 (Cardinality of $H_{\mathcal{F}}$)**.** *If $H^*_{\mathcal{F}}$ is a function on $\mathcal{F}$ satisfying*

$$(1-\lambda)f + \lambda g \preceq (1-\lambda)f' + \lambda g'$$

if and only if

$$(1-\lambda)H^*_{\mathcal{F}}(f) + \lambda H^*_{\mathcal{F}}(g) \leq (1-\lambda)H^*_{\mathcal{F}}(f') + \lambda H^*_{\mathcal{F}}(g') \quad \forall f,g,f',g' \in \mathcal{F}$$

then $H^*_{\mathcal{F}}(f) = aH_{\mathcal{F}}(f) + b$ with $a = H^*_{\mathcal{F}}(f_1) - H^*_{\mathcal{F}}(f_0) > 0$ and $b = H^*_{\mathcal{F}}(f_0)$. $H_{\mathcal{F}}$ is the canonical utility on $\mathcal{F}$ with reference lotteries f_0 and f_1.

Proof. From Equation 9, we have by hypothesis on $H^*_{\mathcal{F}}$ and $\lambda = H_{\mathcal{F}}$

$$\begin{aligned} H^*_{\mathcal{F}}(f) &= (1-\lambda)H^*_{\mathcal{F}}(f_0) + \lambda H^*_{\mathcal{F}}(f_1) \\ &= (1-H_{\mathcal{F}}(f))H^*_{\mathcal{F}}(f_0) + H_{\mathcal{F}}(f)H^*_{\mathcal{F}}(f_1) \\ &= \left[H^*_{\mathcal{F}}(f_1) - H^*_{\mathcal{F}}(f_0)\right] H_{\mathcal{F}}(f) + H^*_{\mathcal{F}}(f_0) \end{aligned} \tag{10}$$

The last line implies that $a = H^*_{\mathcal{F}}(f_1) - H^*_{\mathcal{F}}(f_0) > 0$ since $f_0 \preceq f_1$ by assumption. This establishes cardinality of the Knightian utility index $H_{\mathcal{F}}$ on the lottery space $\mathcal{F}$ and thus completes the proof. □

A.3 Proof of Proposition 3

Continuity, symmetry and additivity simply carry over from the underlying functions, ln and the summation. The maximality statement follows from directly solving the associated optimization problem. Explicitly, we have

1. The natural logarithm is a continuous mapping from the positive reals into the positive reals, $\ln : \mathbb{R}^+ \to \mathbb{R}^+$
2. The ln and sum functions are both commutative
3. Solving the optimization problem

   $$\max H^n_\alpha(s) \quad \text{subject to} \quad \sum_{i=1}^{n} s_i = 1$$

   leads to a maximum for $s_1 = s_2 = \ldots = s_n = \frac{1}{n}$ and this maximum value depends only on the number of possible states of the world because of $H^n_\alpha(\frac{1}{n}1^n) = \ln n$. Hence, the codomain of H^n_α is $\{0 \leq H^n_\alpha(s) \leq \ln n\}$
4. $\alpha > 0, \quad \alpha \neq 1$: proof by insertion.

   $\alpha = 1$: define $0 \ln 0 := 0$, then proof by insertion.
5. This is a consequence of $\ln(ab) = \ln a + \ln b$, $\forall a,b \in \mathbb{R}$.

□

A.4 Proof of Proposition 4

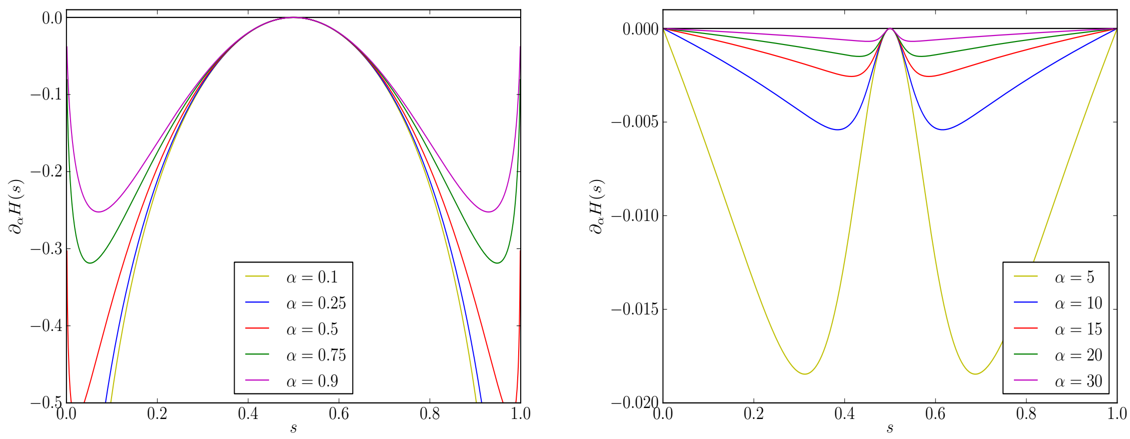
By direct calculation, we obtain

$$\frac{\partial}{\partial \alpha} H^n_\alpha(s) = \underbrace{\frac{\ln \sum_i s_i^\alpha}{(1-\alpha)^2}}_{\text{I}} + \underbrace{\frac{1}{1-\alpha} \frac{\sum_i s_i^\alpha \ln s_i}{\sum_i s_i^\alpha}}_{\text{II}}$$

Case-by-case analysis shows that the terms I and II behave in opposite directions for $0 < \alpha < 1$ and $\alpha > 1$, respectively. When $\alpha > 1$, term I is negative for any s due to $\sum_i s_i^\alpha < 1$ whereas term II is positive since it is a product of two negative numbers. On the other hand, in the case $0 < \alpha < 1$, we find that I>0 while II<0. The overall behavior of $\frac{\partial}{\partial\alpha} H_\alpha^n(s)$ is thus not evident from these considerations. Without loss of generality, we consider the special case $n = 2$, so that

$$\partial_\alpha H_\alpha^2(s) = \frac{\ln(s^\alpha + (1-s)^\alpha)}{(1-\alpha)^2} + \frac{1}{1-\alpha} \frac{s^\alpha \ln s + (1-s)^\alpha \ln(1-s)}{s^\alpha + (1-s)^\alpha}$$

From Figure 2, we see that for every non-degenerate lottery, i.e. $s \neq 0{,}1$, $\partial_\alpha H(s)$ is



(a) $0 < \alpha < 1$

(b) $\alpha > 1$

Figure 2: The behavior of $\partial_\alpha H_\alpha^2(s)$ for different shares $s_1 = s$ so that $s_2 = 1 - s$. The curve remains in the negative codomain for any valid s.

negative. A numerical investigation of the case $s = 0.5$ reveals that $\partial_\alpha H(s)$ comes very close to zero from below but without touching it. The proof for $n > 2$ goes accordingly.

□

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