Didactics of Mathematics in Higher Education as a Scientific Discipline - Conference Proceedings

Göller, Robin; Biehler, Rolf; Hochmuth, Reinhard Karl; Rück, Hans-Georg

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Conference Proceedings

Editors:

Robin Göller, Rolf Biehler, Reinhard Hochmuth, Hans-Georg Rück

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Preface

Mathematics education at the tertiary level is a practical concern in many institutions of higher education, and efforts are being made world-wide to improve its quality. A growing number of mathematicians and mathematics educators see the need for doing research and thoughtful development work in mathematics education not only at school level, but also at tertiary level. To give momentum to the establishment of a scientific community of mathematicians and mathematics educators whose concern is the theoretical reflection, the research-based empirical investigation of mathematics education at tertiary level, and the exchange of best- practice examples, the khdm (German Centre for Higher Mathematics Education, www.khdm.de) and the Volkswagen Foundation jointly organized a conference named "Didactics of Mathematics in Higher Education as a Scientific Discipline", which was held from 1st to 4th December 2015 in Hannover, Germany, at Schloss Herrenhausen. We are delighted that about 100 experts from 16 different countries with scientific background in mathematics or mathematics education followed our invitation to present and to discuss research and innovative efforts for improving the teaching and learning of mathematics at tertiary level, as well as experiences from teaching practice and empirical and theoretical research approaches that aim at a better understanding students' difficulties in learning mathematics and in learning to think mathematically.

We are very grateful to the Volkswagen Foundation for providing full financial support for this event and for providing the conference venue Schloss Herrenhausen. Without the Volkswagen Foundation and the Stiftung Mercator the khdm would probably not exist. In 2009, both foundations made a call for proposals for creating subject centers for university education. Fostering excellence also in university teaching and not only in university research became a big issue in Germany since then. The call of the foundations piloted these developments. In this competition, the universities of Kassel and Paderborn were successful with their application for a center for higher mathematics education, the khdm. The universities Kassel and Paderborn are geographic neighbors and had good collaborations already in several domains and the fact that Rolf Biehler moved from Kassel University to Paderborn University in 2009 was also supportive.

The idea of the two foundations was to provide funds for a starting phase of about three years. Afterwards the universities were supposed to take over the center and maintain it with own funds and with third party funds, in case it were successful. The idea was then to attract further researchers from inside and outside these universities to join the khdm with their own projects, their Ph.D. and post graduate students to make it grow. This happened to a large extent. The khdm is institutionalized as a joint scientific institute of the universities of Kassel, Paderborn and Lüneburg since 2012. Lüneburg joined in 2012 when Reinhard Hochmuth moved from Kassel to Lüneburg University. The next step is to extend the khdm to the University of Hannover, where Reinhard Hochmuth moved to in 2014. There are already several ongoing projects together with researchers from the University of Hannover.

When the khdm started in the fall of 2010, the Mercator Stiftung and the Volkswagen Foundation financed five full positions and the universities financed one scientific center manager and half a secretary. The proposal was supported by a team of 15 professors from mathematics, mathematics education, psychology and didactics of university education. Today, the khdm has more than 50 members, and among them there are about 15 Ph.D. students. About 30 khdm members were participants of this conference and presented their research.

The original objective for creating the khdm was to build a center that will support community building in university mathematics education and related educational research on a national and international level, that will design and perform research and development projects in university mathematics education and that will contribute to the emergence of "Didactics of Mathematics in Higher Education as a Scientific Discipline". This is the title we chose for this conference.

Members of the khdm are presenting their research internationally e.g. at PME, CERME, RUME, INDRUM and ICME conferences. In addition, we were able to convince the Volkswagen Foundation to provide additional "last" funds for supporting the international networking in a moment where the khdm has become adult and has to leave the nursery provided by the foundations. We are very grateful to the Volkswagen foundation that the khdm could present its work to an international audience and that on the other hand the khdm was encouraged and supported to invite researchers from Europe and abroad for presenting their work for our mutual benefits, for refreshing scientific and personal relationships and for creating new ones.

The scientific program of the conference was structured into nine working strands, which are specified below. Besides, on each of the four days of the conference there was a keynote talk. Barbara Jaworski provides an overview of the study and development of teaching at university level, involving both research projects and projects largely of a developmental nature. Considering a range of theoretical perspectives underpinning research studies and studies, which focus on innovations in teaching, pointing particularly to the issues they raise for teachers and the wider community she concludes with a vision of developmental research which enhances knowledge in practice as well as contributing to knowledge in the scientific community. Rolf Biehler and Reinhard Hochmuth use concepts from the Anthropological Theory of Didactics for a characterization of so-called mathematical bridging courses with view of the praxeologies they are supposed to prepare for or to bridge into. The characterization takes into account the variety of study programs at the university and the diversity of goals and relations to previous school mathematics. Chris Rasmussen expands the constructs in Cobb and Yackel's interpretive framework that allow for coordinating social and individual perspectives to contribute to the coordination of different analyses to develop a more comprehensive account of teaching and learning. Finally, Aiso Heinze discusses theoretical conceptualizations and empirical studies of teachers' mathematical content knowledge and suggests a conceptualization of the content knowledge needed for teaching secondary mathematics.

Aside from the keynote talks, there were oral presentations of different length as well as poster presentations and time slots for comprehensive discussions. Each of these presentations was allocated to one of the nine working strands, although many presentations would match several strands. The present proceedings adopt this structure. The nine working strands are:

1. Mathematics as a subject in pre-service teacher education

Educating future school mathematics teachers at university poses specific challenges with regard to the "mathematics for teaching" that is necessary from the perspective of their future profession and learnable by teacher students. Working on the "double discontinuity" (Felix Klein) is part of this challenge.

2. Mathematics for math majors

This working strand addresses the specific concerns regarding the teaching and learning of mathematics for math majors. Research on teaching and learning topics such as Analysis, Linear Algebra, Abstract Algebra and Differential Equations are part of this session as well as other topics from undergraduate and graduate education. Transition to advanced mathematics courses such as transition-to-proof courses or calculus courses with a perspective to analysis courses are part of this strand.

3. Mathematics as a service subject (in engineering and economics)

This working strand addresses the mathematics education of students within a non-math major, and focuses on engineering, economics, natural science, etc. It aims to exchange and discuss approaches for better connections between mathematics and the major subject, difficulties arising due to different meanings of mathematical concepts in (subject-specific) applications and in institutional practices.

4. Tertiary level teaching (analyses, support and innovations)

Studying the practice of teaching of professors, lecturers and teaching assistants is one focus of this strand, as are programs for supporting different kinds of university teachers. A second focus of this strand is concerned with innovative teaching methods such as e-learning, blended learning, flipped classroom approaches lectures with cognitively activating elements etc.

5. Motivation, beliefs and learning strategies of students

This working strand focuses on students' motivation, attitudes and learning strategies as an important condition for successful and deep learning. Studies that focus on studying the development of student beliefs and working methods were welcome as well as studies that aim at influencing student engagement directly.

6. Learning and teaching of specific mathematical concepts and methods

This working strand focuses on the teaching and learning of specific mathematical concepts (e.g. convergence, derivative, groups) and on practices that are specific to mathematics (e.g. proving, reading and writing mathematical texts) which are known to be difficult for students to understand and learn. Strand 6 provided a place for discussing theoretical approaches to analyzing the teaching and learning of such concepts and methods, and to an exchange of best-practice examples to overcome these difficulties.

7. Curriculum design including assessment

This strand focuses on whole courses or large parts of them that are reflected and redesigned from the perspective of consciously considered competence goals. Strand 7 provided a venue to discuss innovative lectures designed to address students' transition problems or, to introduce students to mathematical thinking and learning, as well as discussions of traditional courses that are restructured so that assessments are better aligned to the goals of the course (constructive alignment). Studies in the perspective of design-based research were particularly welcome.

8. Theories and research methods

This strand focuses on presentations and (critical) reflections of theories and research methods used for research in tertiary mathematics education. This comprises theoretical or methodical frameworks (e.g. for data analysis), models of quantitative and qualitative research or considerations about phrasing and testing goals, competencies, personality traits. Theories may include institutional approaches (ATD), sociocultural approaches as well as cognitive-epistemological theories of mathematics and its learning.

9. Transition: research and innovative practice

This strand focuses on theoretical analyses of the transition problem, as well as on approaches, courses, or support structures designed to overcome difficulties that students experience in transitioning from secondary to tertiary mathematics. This comprises mathematical bridging courses before the first semester or remedial course in the first semester as well as new elements in the teaching of first year university courses that take the transition problem into account, such as mathematics support centers.

Kassel, Paderborn, Hannover, January 2017

Robin Göller, Rolf Biehler, Reinhard Hochmuth, Hans-Georg Rück

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KEYNOTE TALKS

Relating different mathematical praxeologies as a challenge for designing mathematical content for bridging courses*

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This contribution applies concepts from the Anthropological Theory of Didactics (ATD) to an ideal-typical characterization of so-called bridging courses in view of their primary goals. Our considerations are illustrated by discussing mathematical content for bridging courses. We are convinced that such a systematization might be helpful for designing and optimizing specific mathematical content that relates to the different mathematical praxeologies represented in the variety of study programs at the university.

The problem of bridging: From where and into what?

Our contribution aims at a more systematic and theoretical description of so-called bridging courses by making use of basic concepts from the Anthropological Theory of Didactics (ATD). ATD has been already used by the second author of this paper for analyzing several problem domains in university mathematics education (Hochmuth & Schreiber 2015a, b; Hochmuth 2016). This genuinely joint paper – expressed by the alphabetic order of the authors – extends this approach for the first time to bridging courses, where the two authors build on many joint discussions and collaborative material development experiences as co-leaders of the VEMINT-project where blended learning bridging courses have been designed, evaluated and improved since 2003 (see e.g. Biehler, Fischer, Hochmuth & Wassong, 2012). By bridging courses we understand courses that are offered to future students who have just finished school. At least in Germany, bridging courses have been established at all universities during the past years. They represent one answer beside others to the well-known transition problems from school to university concerning in particular mathematical knowledge and competences. Often universities offer a variety of bridging courses adapted to different study programs, for example courses for mathematics majors, for future secondary teachers, mathematics courses for economic or engineering studies, and their different mathematical knowledge requirements. Bausch et al. (2014) offer an overview of current courses and their different rationales. Most of these courses are offered before the first semester starts and last 2 to 6 weeks but there is a growing number of courses within the first semester that follow the idea of bridging between school mathematics and university mathematics, for instance the course "introduction into the culture of mathematics" (Biehler & Kempen, 2015; Kempen & Biehler, 2015). The situation of bridging is more complex with regard to students who intend to become teachers (see also Bessenrodt et al., 2015): the school mathematics culture should not just be replaced by the university mathematics cul-

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ture, as the teachers will re-enter the school culture again after their studies so that a second bridge might be necessary. This is a well-known problem for which Felix Klein (1933) coined the famous notion of the "double discontinuity" in the studies of teacher students. We consider Klein's book(s) also as a "bridging course", however a bridging course placed at the end of a study program for future mathematics teachers, where students are already acquainted with the university mathematics culture.

In the VEMINT project, we have developed three versions of bridging courses, one for mathematics majors including future Gymnasium teachers, one for engineering studies, and one for primary and lower secondary (middle school) mathematics teachers. The differences of these courses primarily reflect the differences of the cultures they are supposed to bridge into and not so much assumptions about differences with regard to mathematical competencies of the three different student populations. Everybody will agree that the mathematical practices in courses for mathematics majors will differ from those of mathematics for engineers, although a clear theoretical analysis of the differences has not been done so far. We think that ATD offers an adequate theoretical framework for doing this. At least in Germany, we can also observe a third mathematical culture for primary and lower secondary teacher students that is again different. We have specific series of books and lecture notes for these students (often developed by researchers in mathematics education who are responsible for these courses), where this culture in general is "closer" to the school mathematics culture: more visual representations, more application, more motivations and explicit relations to school mathematics, intuitive kinds of reasoning, reflective elements, more mathematics as a process than mathematics as a ready-made product etc. These differences are differences as compared to the culture of mathematics for mathematics majors. On the other hand these courses also differ from school mathematics in various aspects, such as the preciseness of concept definitions, the role of proof and systematic theory development. In a sense this culture can be reinterpreted as a bridging culture itself between school and the university mathematics for mathematics majors. That is why we built on this culture when we were designing bridging course materials in the VEMINT project for ALL students.

A good case in point for such books are Kirsch (2004) and Müller, Steinbring & Wittmann, 2004). Last but not least, the culture of school mathematics is also not homogeneous, most federal states in Germany distinguish basic from advanced level in their university bound school courses, and there is evidence that the relative importance of techniques, technology and theory in the sense of ATD is different and not just the quantity of content.According to those different groups of studies bridging courses follow up different goals representing a different understanding about a helpful bridge between school and university mathematics. We will try to make distinctions after we have introduced some notions from ATD.

Some notions from ATD

ATD (Chevallard, 1992, 1999; Winslow, Barquero, Vleeschouwer & Hardy 2014) aims at a precise description of knowledge and its epistemic constitution. Its concepts allow explicating institutional specificities of knowledge and related practices. Behind this approach is the conviction that cognitive-oriented accesses tend to misinterpret contextual or institutional aspects of practices as personal dispositions. A basic concept of ATD are praxeologies, which are represented in so called "$4T$-models $(T, \tau, \theta, \Theta)$" consisting of a practical and a theoretical block. The practical block (know how, "doing math") includes the type of task (T) and the relevant solving techniques (τ). The theoretical block (knowledge block, discourse necessary for interpreting and justifying the practical block, "spoken surround") covers the technology (θ) explaining and justifying the used technique and the theory (Θ) justifying the underlying technology. Praxeologies give descriptions of mathematics by reference models that are activity oriented (techniques, technologies). The interconnectedness of knowledge is modelled by ATD by means of local and regional mathematical organizations that allow contrasting and integrating practical and epistemological aspects in view of different institutional contexts. Therefore ATD is in particular helpful in analyzing mathematical knowledge and its different institutional realizations within different learning contexts.

Towards a Praxeological Characterization of Different Bridging Courses

We will concentrate on bridging courses that take place between finishing school and beginning university courses and aim at bridging the praxeologies at school and university level. It is clear that the mathematical competences of students after school are not the same but rather heterogeneous. Therefore bridging courses have to offer materials and learning situations that fit to very different competence prerequisites. In the following we blind out such variations and related questions concerning didactical processes and consider "simply" knowledge in the institutional perspective of school and university studies formulating abstract reference models in the sense of ATD. Of course, designing courses requires taking into account further ideas that blend with ATD, for example the subject scientific approach as discussed in Hochmuth & Schreiber (2015). This also means that the existing bridging courses show much more variance than reflected in our abstract distinctions.

Following Winslow & Grønbæk (2013) we refer to the notion $R_I(x, o)$ introduced by Chevallard, (1991) to indicate the relation of a position x (roles of persons such as teachers and students) within an institution I to a praxeology o. We will consider in the following three institutions: school (S), university (U) and the transition from school to university that is in the following represented by an arrow as well as by the diagrams as such. Within the institution school the position x is given by the school student s and within university by the student σ. In ATD-terms the transition from school to university can then be noted by

$$R_S(s, o) \rightarrow R_U(\sigma, \omega),$$

where o represents a praxeology within school and ω some praxeology within university.

The mathematical praxeologies ω of different study programs are different as such but also with regard to what components of school mathematical praxeologies are relevant for their own praxeologies. Techniques $\tau(o)$, technologies $\theta(o)$ or theories $\Theta(o)$ can be differently relevant. For instance, maths in engineering courses will require routine skills in techniques for calculating derivatives and integrals, and may direct a bridging course to make sure that these skills are active knowledge of their beginning students, some new tasks and corresponding techniques $\tau(o)$ maybe added that are needed in engineering classes. They could have been part of the school curriculum but have been deleted in recent curriculum reforms (for instance logarithms as a function). Math major bridging courses may wish to point out the gaps and problems with the theoretical foundation in school mathematics as regard to the concept of differentiability or the need for a precise limit concept for defining derivatives. But engineering courses may also wish to make the technologies $\theta(o)$ to a topic, as the requirements in engineering math contexts may include a more profound knowledge with regard to the application conditions of techniques that are used.

Ideal types of bridging courses

Type A: Improving skills in applying techniques steming from current or past school mathematics

We can symbolize the bridging process as $R_S(s, o) \rightarrow R_U(\sigma, \omega[\tau(o)])$.

Examples include techniques for solving quadratic and exponential equations, for manipulating terms with fractions, roots and trigonometric expressions. The course can go beyond school mathematics in adding new tasks and new techniques that are relevant for the future university courses. We know that school mathematics is split into many different local or regional mathematical organizations. The bridging course can aim at relating these organizations and systematize them. For instance, a course can have a chapter on „Solving equations“, where the different types of equations of 12 years schooling are systematically related to each other. This systematization may touch the level of technologies. But in general, these courses do not profoundly change the technology and the theory level. In this precise sense they remain completely on the level of school mathematical praxeology, although they may extend and add tasks and techniques.

Type B: Improving technical skills and technological competences in school mathematical contexts

The transition we mean can be symbolized as $R_S(s, o) \rightarrow R_U(\sigma, \omega[\theta(o)])$.

Type A courses often focus on sets of skills that in principle can be performed by computer algebra systems. However, also the mathematical practice of engineers requires a deeper knowledge about the technology of the techniques used: What are the conditions where techniques can be applied? What are the limitations of techniques? What is the efficiency of a certain technique? Declarative technological knowledge has to be strengthened. For instance, what type of technique can be used with a certain type of equation (simple algebraic manipulations, solution formulas, numerical or graphical solutions). Moreover, these courses may be also based on a didactical assumption, namely that learning techniques by heart without understanding the underlying technology is often not a sustainable investment of time. For instance, a bridging course may raise the question why or in which sense the rule for adding fractions is “true” and which arguments can be out forward against the “simpler” rule

$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$. Or, students may have to become aware that the rule $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

was explicitly introduced at school level for natural numbers $a, b, c, d,$ whereas it has to be used in university mathematics for all real numbers $a, b, c, d,$ which can be expressions with roots and fractions itself. It is a real didactical challenge to "justify" this general use of the latter formula. We know of no school or university book that makes this a big topic, but it may become part of the technological knowledge communicated in a bridging course.

Type C: Introducing theoretical and technological aspects of university mathematical practice within topics from school mathematics

An example topic in such a course can be the function concept or the divisibility of natural numbers. The notion of function that students bring with them to the university does not pay much attention to domain and codomain, it is often bound to representations of function rules by simple formulas that must contain a variable x. Properties of functions such as intervals of monotonic growth are "seen" in "the" graph etc. Bridging courses as the math major version of VEMINT introduce university definitions and new tasks and techniques related to domain, codomain, injectivity and bijectivity etc. Simple proofs and theorems are formulated according to the university mathematics culture. However, much more time and explanation is provided as compared to a standard university lecture.

Another topic can be divisibility of numbers, where the course can start with developing general proofs for seemingly simple statements such as that the sum of three consecutive natural numbers is always divisible by three (the course Kempen & Biehler (2015) refer to) up to a little theory of rules for divisibility and their justification (Hilgert & Hilgert, 2012). Gueudet (2008) also suggested such topics as domain for activities on the level of the new university mathematics culture. Grieser (2013) follows this approach on similar topics and concepts of school mathematics but in a sense that emphasizes mathematics as process of problem solving and proving in the sense of Polya. A symbolization for this type of bridging course could be $R_S(s, o) \rightarrow\ R_U(\sigma, \omega[\Theta(o)])$, meaning that a theory and technology on the level of university mathematical practice is provided for familiar objects of school mathematics, however the tasks and techniques related to these objects are largely transformed.

Type D: Reflecting relations between school and university mathematics

Type C courses may help students coping with the transition because such courses reduce „cognitive load" and can enhance self-efficacy and confidence in coping with the new culture. The new culture is introduced at a reduced speed for familiar objects. However, from the perspectives of future Gymnasium teachers this may not be enough. Reflective elements seem to be necessary that explain the reasons for the new culture and make the differences much more explicit than the reflective elements of a usual type C course can do in limited time.

This would be something that Felix Klein had in mind when he wrote his book. But let us rely on a metaphor. We can regard a type C course as a compact language course where not much reflection between the old language and the foreign language can be done, but the new language is practiced under favorable conditions before giving access to the foreign language culture where the students have to survive themselves. A profound reflection on the difference and relation between both cultures will be better possible after one has become a part of the new culture. Therefore a second type of bridging course might be placed at the end of a study program for teachers. Such courses may contain „interface tasks" in the sense of Thomas Bauer (2013) or may be part of courses on didactics of mathematics, such as Danckwerts & Vogel (2006) on teaching and learning calculus, which, among others should make the reflection between the different cultures a topic of their curriculum.

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Teachers' mathematical content knowledge in the field of tension between academic and school mathematics

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The question as to what content knowledge mathematics teachers need is highly relevant for the design of education programs for mathematics teachers. Hence, scholars have been trying to find answers to this question for a long time. Corresponding theoretical conceptualizations and empirical studies of teachers' mathematical content knowledge diverge widely, however. – This is particularly valid with respect to whether the construct is oriented more towards school mathematics or towards academic mathematics. In this presentation, we will discuss different theoretical as well as empirical approaches and suggest a conceptualization of the content knowledge needed for teaching secondary mathematics. Furthermore, results of an empirical study with 505 pre-service teachers will be presented.

Introduction and Theoretical Background

Empirical findings indicate that professional knowledge of mathematics teachers contributes to instructional quality and to student learning (e.g., Krauss et al., 2008; Hill, Schilling, & Ball, 2005; Hill et al., 2008). Consequently, there is a consensus that professional knowledge is a key goal of teacher education. Models of teachers' professional knowledge consider content knowledge (CK) and pedagogical content knowledge (PCK) as important components (Shulman, 1986; Baumert et al., 2010). Though CK and PCK are directly addressed in courses of teacher education programs at university, the development of teachers' professional knowledge is still not comprehensively understood. In particular, there is a lack of longitudinal studies that analyze how CK and PCK integrate to domain-specific teacher knowledge.

One of the main challenges for research on teacher education lies in an adequate modeling and assessment of domain-specific knowledge. For the subject mathematics, some standardized tests of teachers' CK and PCK already exist. However, these existing approaches differ widely in the way they operationalize CK and PCK. In particular, in the case of CK the existing tests range from mathematical knowledge as it is taught in school to knowledge as it is considered in first semester courses in teacher education programs. In our study KiL (*Measuring the professional knowledge of pre-service mathematics and science teachers*, Kleickmann et al., 2013), we also developed instruments for the assessment of mathematics teachers' professional knowledge and focused especially the component CK. We started our consideration from the question which type of mathematical knowledge teachers need for teaching mathematics. In Germany, the mathematics program for pre-service teachers for the upper secondary level is similar to that of undergraduate students majoring in mathematics. Accordingly, we analyzed how CK acquired in undergraduate courses on scientific mathematics can become effective in a school context and how this kind of CK can be assessed.

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In the following, we (1) review the state of research on (pre-service) teachers' CK and PCK, (2) argue for the need of distinguishing content knowledge (CK) from school-related content knowledge (SRCK) and describe tests for CK, SRCK, and PCK, and finally (3) present empirical results on the structure of teachers' domain-specific knowledge.

The Constructs CK and PCK in Recent Research

Following the idea of Shulman (1986), the constructs CK and PCK were operationalized in several empirical studies investigating mathematics teachers. Although studies were able to show the importance of the assessed knowledge components for teaching quality and student learning (e.g., Baumert et al., 2010), they could not answer the important questions concerning the structure of mathematics teachers' knowledge. In particular, the relation between CK and PCK is still unclear: though these components are clearly separable from a theoretical point of view, most studies found that CK and PCK are highly correlated and sometimes even hard to separate (Hill et al., 2004, 2005; Krauss et al., 2008; Blömeke, Kaiser, & Lehmann, 2008). However, it is not clear if this strong correlation is caused by the underlying conceptualizations, the different operationalizations or if it mirrors the nature of the investigated cognitive structures. Although CK is often described as knowledge on scientific mathematics acquired through formal teacher education, most operationalizations are predominantly focused on mathematical school content. This means in particular that the corresponding tests are not appropriate to measure learning progress in pre-service teacher education. Similarly, PCK is described as a kind of knowledge specific for teaching mathematics but existing test items are often solvable by analytical mathematical competences so that the delineation is difficult (Buchholtz, Kaiser, & Blömeke, 2014).

Regarding the conceptualization of PCK, we follow the suggestions of the COACTIV study and consider three components: knowledge of instructional strategies for a certain topic, knowledge about student cognitions, and knowledge about the learning potential of mathematical tasks (Baumert et al., 2010). Following Shulman we understand PCK as the knowledge "which goes beyond knowledge of subject matter per se to the dimension of subject matter knowledge *for teaching*" (Shulman, 1986, p. 9, emphasis in original) and suggest a rigorous operationalization in this sense. This means in particular that test items do not have a predominant mathematical demand and cannot be solved by mathematical means (e.g. a mathematical argumentation; cf. Buchholtz, Kaiser, & Blömeke, 2013).

In our study, CK was conceptualized as *academic* mathematical knowledge, as it is presented in mathematics courses in formal teacher education. In this conceptualization CK is of a similar type as the type students majoring in a mathematics program are acquiring. Hence, it is clearly beyond school mathematics and our conceptualization of CK is not restricted to elementary mathematics from a higher viewpoint (Klein, 1908). Instead, we follow the original idea of Shulman (1986) who wrote that "subject matter understanding of the teacher [to] be at least equal to that of his or her lay colleagues, the mere subject matter major" (p. 9). It is clear that the programs for pre-service secondary teachers and students majoring in mathematics differ in the number of mathematics courses and also in the specialization of the content. Accordingly, we restricted the expected CK of pre-service teachers to the courses in a bachelor mathematics program. This encompasses the introductory courses (e.g. analysis, linear algebra) as well as specific topics of advanced courses (e.g., classical and modern algebra) which provide a deeper understanding of the mathematical content in school.

School-related content knowledge as applied content knowledge for teaching

If we conceptualize CK and PCK as presented in the previous section, then specific aspects of mathematical knowledge for teaching are missing. Both conceptualizations do not encompass knowledge about school mathematics and curricular knowledge (aspects which are considered as important by Shulman, 1986, or Hill et al., 2005). Moreover, beyond the content and its sequencing, teachers are faced with two additional mathematical challenges which influence their instruction and which originate from the non-trivial relation between school mathematics and academic mathematics. First, teachers must be able to reduce and simplify academic mathematical content so that it is accessible for students on a certain age level (cf. "unpacking mathematics", Ball & Bass, 2003). For example, in German schools in grade 9 rational numbers are extended to real numbers. Teachers should be aware that a construction of real numbers via Cauchy sequences or Dedekind cuts is not accessible for grade 9 students. In contrast, the idea of approximating irrational numbers with the help of nested intervals is feasible. Second, teachers must know how topics of school mathematics are rooted in academic mathematics. For example, to understand the (non-trivial) validity of $0.999... = 1$ which occurs in grade 6 in German schools, teachers must be able to understand $0.999...$ as a geometric series which converges. The understanding that a limit process plays a significant role in this case leads the teachers to possible obstacles for students learning and helps her/him to analyze students arguments for or against the identity $0.999... = 1$.

Summarizing the previous information, it becomes clear that there is a need for a construct related to teachers' mathematical knowledge which is not CK or PCK (as conceptualized in the previous section). This construct encompasses a specific type of mathematical content knowledge, namely a type of content knowledge applied in a school context for the teaching purpose. We denote it as school-related content knowledge (SRCK). The idea that teachers' CK must be more than academic content knowledge and has to be complemented by a kind of applied knowledge was already discussed decades ago. Due to space limitations we just mention the reflections on the profession of mathematics teachers and on the relation between academic mathematics and school contents from the 1970s and 1980s (cf. metamathematics, e.g. Fletcher, 1975, Dörfler & McLone, 1986; cf. mathematical background theory, e.g. Vollrath, 1988).

Investigating CK, SRCK and PCK of Pre-service Mathematics Teachers

Following the idea of three dimensions of domain-specific teacher knowledge (CK, SRCK, and PCK), we developed a test instrument (see Figure 1 for sample items). For the item development, we conducted a curricular analysis of teacher education programs of different universities and curricula for school mathematics (both for secondary level, i.e. grades 5-13, in Germany). In total, we obtained 118 items (PCK: 31, SRCK: 34, CK: 54) that showed adequate psychometric properties in a pilot study. The items were bundled in two test booklets, one test booklet for pre-service mathematics teachers for the academic track, the other for pre-service mathematics teachers for the non-academic track. The booklets had a considerable overlap of 81 items in order to allow a linking of the data on a common scale using IRT.

The tests covered topics from arithmetics/algebra, analysis, geometry, stochastics, and numerics with a strong focus on arithmetics/algebra. According to the curricular analysis the test items cover the characteristics of university-based teacher education sufficiently. The testing time was 120 minutes for each booklet. The items were scored according to a scoring rubric with partly dichotomous, partly partial scores (0, 0.5, 1). The 34 open answer items were scored by two independent raters and the interrater-reliability was considered as sufficient since Cohen's Kappa was above κ =.73 for all items.

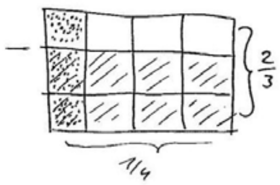
PCK	Please sketch an iconic illustration to explain to a 6th grade student how to multiply a fraction by another fraction. Take ${}^{1}/_{4} \cdot {}^{2}/_{3}$ as an example.	Sample Answer (scored as correct)  "Only the areas marked as  count. How many are these? How many are there in total? ${}^{2}/_{12} = {}^{1}/_{4} \cdot {}^{2}/_{3}$ "
SRCK	Which of the following tasks enable students to discover that rational numbers are dense in real numbers? *apt* / *wrong* Measure the length of the diagonal of a square (length 10 cm). ☐ × Seek the smallest fraction bigger than $\sqrt{2}$. × ☐ Split 100€ into three equal portions. ☐ × Find ten fractions between $\sqrt{2}$ and $\sqrt{3}$. ☐ ×	
CK	We call a field extension $L{:}\,K$ finite-dimensional if $dim_K L < \infty$. Which of the following field extensions are finite-dimensional? *right* / *wrong* $\mathbb{Q}(\pi){:}\,\mathbb{Q}$ ☐ × $\mathbb{Q}(i){:}\,\mathbb{Q}$ × ☐ $\mathbb{C}{:}\,\mathbb{R}$ × ☐ $\mathbb{R}{:}\,\mathbb{Q}$ ☐ ×	

Figure 1: Sample items for PCK, SRCK and CK.

Sample and Methods

The sample of the study comprised N = 505 pre-service mathematics teachers from different German universities. On average, the students were 23.3 (SD = 2.9) years old and in their 5.9 semester (SD = 2.64). About 64% of the students aimed to teach in academic track schools (German Gymnasium). The data was modeled by a multidimensional random coefficients multinomial logit model (MRCML; Adams, Wilson & Wang, 1997) in order to examine the structure of pre-service teachers' knowledge. In total, 98 items satisfied the required cutoffs for item quality indicators and were included in the model.

Results

To examine the separability of the constructs CK, SRCK and PCK, we contrast a three-dimensional model against a one-dimensional model (g-factor model). Moreover, since SRCK can be considered as knowledge related to CK as well as to PCK, we also include two two-dimensional models combining SRCK with CK and with PCK respectively (see Table 1). We used the Bayesian information criterion (BIC) to compare the fit of the different models. Here, smaller values indicate a better model fit and a difference greater than ten is considered as very strong evidence for the model with the lower value (Raftery, 1995, p. 141).

A comparison of the model fit indices given in Table 1 indicates that the three-dimensional model fits the data best, whereas the one-dimensional model shows the worst model fit. The EAP/PV reliabilities of the three scales are good or satisfying (CK: .83 with 41 items, SRCK: .80 with 31 items, PCK: .69 with 26 items). The latent correlation between PCK and CK was estimated as r(PCK,CK) = .54, indicating a good separability of the constructs. At the same time, SRCK correlated highly with both the CK (r(SRCK,CK) = .83) and the PCK (r(SRCK,PCK) = .85) dimension on the latent level. Hence, the construct SRCK cannot be considered as identical to CK or PCK.

Model	Description	n	df	BIC
3D between model	CK – SRCK – PCK	112	44023.82	44720.97
2D between model A	CK/SRCK – PCK	109	44159.14	44837.62
2D between model B	CK – SRCK/PCK	109	44069.37	44747.85
1D general factor model	CK/SRCK/PCK	107	44312.97	44979.00

n = total number of estimated parameters, df = final deviance

Table 1: Comparison of alternate models

Discussion and Outlook

The empirical results provide evidence for a three-dimensional structure of pre-service mathematics teachers' domain-specific knowledge. In particular, school-related content knowledge (SRCK), conceptualized as applying academic mathematical knowledge in the context of school mathematics for teaching purposes, turned out to be separable from CK as academic knowledge although it seems to be deeply rooted in CK. It was also found to be distinguishable from PCK. However, it is an open question whether SRCK as a kind of applied knowledge can be directly taught in teacher education programs on its own or whether it needs academic CK as a consistent and structured foundation. A first step to answer this question is to investigate the longitudinal development of pre-service teachers' mathematical SRCK and to identify the role of CK as influencing factor. In this contribution, we focused SRCK in its relation to CK as academic content knowledge. We want to mention that the presented results can also be interpreted in a way that considers SRCK as a link between CK and PCK (Loch, Lindmeier and Heinze, 2015).

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Teaching mathematics at university level: how we think about teaching and its development

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This paper focuses on the study and development of teaching at university level, involving both research projects and projects largely of a developmental nature. It points to the research and professional literature to acknowledge the existing (and growing) literature base and to contrast studies of different types. It considers a range of theoretical perspectives underpinning research studies and goes on to look at studies which focus on innovations in teaching, pointing particularly to the issues they raise for teachers and the wider community. It concludes with a vision of developmental research which enhances knowledge in practice as well as contributing to knowledge in the scientific community.

Introduction

Mathematics has a very long history. So does mathematics teaching. From sitting at the feet (metaphorically or literally) of the master, to working with the wizziest technology, there are expectations that learners gain from being taught and recognition that teaching can take a wide range of forms. I talk, here, mainly about teaching at the university level, where, it is clear, there are certain *traditions* of teaching and many current *practices*, some of which use the most up to date digital affordances.

In this presentation I intend to address what it means to teach for all those practitioners and researchers for whom the question is important: this includes mathematicians, mathematics educators and mathematics education researchers and of course the students whose task is to learn and make sense of mathematics. I draw on a growing literature which includes research studies and teachers' personal accounts relating to the nature of teaching. I will organise the talk under four headings as follows:

- Traditions and practices
- Theoretical perspectives and constructs
- Pedagogy and Innovation
- Development and research

1. Traditions and practices

In the UK, in a report for the Institute of Mathematics and its Applications (IMA), Hawkes and Savage (2000 p. ii). wrote about "The Mathematics Problem":

> *Evidence is presented of a serious decline in students' mastery of basic mathematical skills and level of preparation for mathematics-based degree courses. This decline is well established and affects students at all levels. As a result, acute problems now*

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confront those teaching mathematics and mathematics-based modules across the full range of universities

Without going into details of the skills and level of preparation expected, this statement suggests a challenge to those teaching mathematics at this level to find ways to help students who are ill-prepared for university-level study in mathematics. In this section, I draw attention to a number of themes, all relating to ways in which mathematics teaching is conceptualised and conducted, with the learning or understanding of mathematics by students as the motivating factor.

a) Traditional approaches to university teaching: the predominance of the lecture (e.g., Pritchard, 2010; 2015)

b) Professional, or pedagogic literature in which university teachers of mathematics expound their own perspectives on organising and teaching a particular topic (e.g., Uhlig, 2003) or offer an approach to a mathematics topic based on a well-defined pedagogic approach (e.g., Burn, 1982; Mahavier, 1999)

c) Research studies which are based in some theoretical perspective (e.g., cognitive or sociocultural theories) or which adopt or develop theoretical constructs to explain the research approach and its findings (e.g., Nardi, Jaworski & Hegedus, 2005).

d) Research studies which seek to illuminate mathematics teaching as it is seen in Higher Education currently (e.g. Hemmi, 2010; Treffert-Thomas, 2015; Weber, 2004); and, in contrast, those which seek to explore innovative approaches to teaching, for example inquiry-based teaching (e.g., Chang, 2010; Jaworski and Matthews, 2011)

e) Research studies which contrast the teaching of mathematics to mathematics students with the teaching of mathematics to students in other disciplines such as engineering. What differences do we, or would we expect to see (e.g., Alpers 2007; Hernandez-Martinez & Harth, 2015; Mokhtar & Rohani, 2010).

A longer paper would delve into the details of these various areas of literature; in this respect see Treffert-Thomas & Jaworski, 2015; Abdulwahed, Jaworski & Crawford, 2012.

2. Theoretical perspectives and constructs

It seems fair to say that theoretical work in relation to *teaching* mathematics in Higher Education is in its infancy. Partly we see theories being adopted and extended from research into teaching and learning at school level. For example, cognitive theories such as constructivism can be found explicitly or implicitly in accounts from research, or from assumptions of those writing about a study: examples include studies which seek to promote students' mathematical constructions (e.g., Mokhtar & Rohani, 2010), or describe teaching approaches as being 'constructivist' in style (e.g., O'Callaghan, 1998). Such studies may bring with them constructs developed at school level within a constructivist base; for example, sociomathematical norms developed by Yackel and Cobb, (1996) are used at undergraduate level by Rasmussen and Kwon (2007). Sociocultural studies can be seen similarly to reflect approaches at school level; for example Jaworski & Potari (2009) characterised mathematics teaching using models from Activity Theory which is used similarly by Jaworski et al, (2012) to characterise teaching mathematics to undergraduate engineering students. Again constructs used to characterise teaching at school level are employed with adaptation in Higher Education, for example, the Teaching Triad, developed by Potari & Jaworski (2002) was used by Jaworski (2002) to analyse tutorial teaching at university level.

The fundamental difference between constructivist and sociocultural theories lies in the origins of *coming to know*. Constructivism focuses centrally on the learning/coming-to-know of the individual: whether this be based within a cognitivist study of the individual making sense of some concepts in mathematics or within a social setting in which the individual responds to communication from others around. In a sociocultural frame, the focus is on how knowledge grows within the social setting and is framed by social factors and issues in the local and wider environments which impose on learning. It is within such social perspectives that the cases described below are situated and we see how the social setting impinges on the ways in which learning is conceived (Lerman 1996; Jaworski 2015).

3. Pedagogy and Innovation

Regarding pedagogy in university mathematics teaching, there is considerable agreement that lecturing is the most common traditional mode of teaching. However, we see disagreement on the nature of lecturing. For example, Wu (1999) speaks enthusiastically in favour of lecturing, whereas Millet (2001) decries Wu's arguments and suggests that lecturing is at the root of students' lack of mathematical understanding. Pritchard, who writes very positively about the value of lecturing, nevertheless acknowledges:

> *[L]ectures are regarded in many disciplines as outdated and ineffective … the fundamental objection is that lectures are essentially transmissive: they are simply a medieval technology for equipping students with slightly inaccurate versions of the lecturer's own notes (Pritchard, 2015, p. 58).*

Perhaps as a consequence of such views, there is a growing literature related to innovation in mathematics teaching at university level addressing ways in which teachers have attended to and thought about the learning of their students and devised innovative approaches to teaching. See for example, Abdulwahed, Jaworski & Crawford (2012) and Treffert-Thomas & Jaworski (2015) which provide examples of new approaches to teaching, promoting learning and experimentation around the ideas on which they are based.

Here, I provide three examples from recent research and development at my own university.

3.1 Engineering students understanding mathematics (ESUM)

This study was conducted within the national HE STEM[1] programme with the aim of supporting first year engineering students' more conceptual understandings of mathematics. It had been noticed that many of these students arrived from their school A-level courses[2] with

[1] HE STEM is a nation-wide programme in Higher Education (in England and Wales) focusing on extending knowledge and practice in Science, Technology, Engineering and Mathematics. http://www.hestem.ac.uk/

[2] 'A level' is a high-stakes national assessment in the final years of secondary schooling (in England and Wales) preparing students for study in higher education.

very procedural understandings, of concepts in basic calculus (*functions* for example) which were seen as central to their higher level studies in engineering mathematics. A team of three experienced teacher-researchers and one research assistant designed an innovation (to promote more conceptual understandings) involving an inquiry-based approach to the teaching, including four elements: inquiry-based questions and tasks, a *GeoGebra* environment for graphical exploration, small group inquiry in mathematics and an assessed small-group project. The innovation was implemented in the first semester of a two-semester first year module with provision of 2 lectures and 1 tutorial per week for 2x13 weeks. The group project was assessed as part of the first semester's activity and the whole module was assessed by traditional examination at the end.

The inquiry-based nature of the innovation included the design and use of mathematical tasks that should draw students into mathematical inquiry and a deeper engagement with mathematics than had been the case previously. It was implemented in tutorials such that students in a small group (n = 4) worked together on given tasks using *GeoGebra* to provide the dynamic graphical environment that the tasks required. For example, students were asked to explore a quadratic function as shown in a part of the task in Figure 1 below. Here task design envisaged students using the graphical environment to find out relationships between lines and curves and to see functions as mathematical objects with a range of properties.

Task 1

1b) If we have the function $f(x) = ax^2 + bx + c$.
What can you say about lines which intersect this function twice?

1c) Write down equations for three straight lines and draw them in GeoGebra
Find a (quadratic) function such that the graph of the function cuts one of your lines *twice*, one of them *only once*, and the third *not at all* and show the result in GeoGebra.
Repeat for three *different* lines (what does it mean to be different?)

Figure 1: A typical inquiry-based task

The study was conducted from a sociocultural perspective in which the wider setting and its influences on learning and teaching were significant to analyses of data. Data were collected from observations of lectures and tutorials, lecturer oral and written reflections, deliberations of the teaching team; surveys of student factors and perspectives, and post-teaching focus-group interviews with students. While scores on a traditional style exam were on average 10% higher than for previous cohorts, students' focus-group views on the module indicated that though they understood the purposes of the intervention they nevertheless would have preferred a more traditional approach to the module. Analyses drew attention to ways in which cultural and systemic factors (e.g. student cultures; university systems) dealt in complex ways with student cognition in making sense of mathematics within the inquiry-based approach. An activity theory analysis enabled us to make sense of these findings (Jaworski & Matthews, 2011; Jaworski, Robinson, Matthews & Croft, 2012).

3.2 Group work in mathematical modelling

This study also involved teaching mathematics to engineering students, this time in a one-semester second year module in which the innovation involved activity of students in small groups (4 to 5), using mathematical modelling tasks, as a complement to traditional style lectures. The research question guiding analysis was: *How do social interactions in a small group collaborative work influence the students' mathematical sense making and the outcome of the activity?* (Hernandez-Martinez & Harth, 2015).

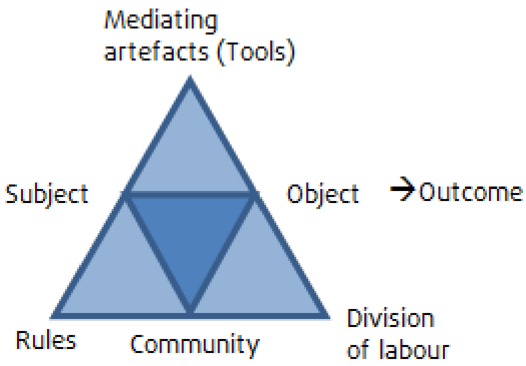

Figure 2: The CHAT framework

The study was based socioculturally, using as a framework for analysis *Cultural-Historical Activity Theory* (CHAT), which draws attention to the complexity of (social) factors mediating human activity, in this case collaborative learning in a mathematical modelling task. Modelling tasks were designed to address mathematical topics within the module, such as ordinary differential equations, and used in tutorials with groups of students. Data from observations of the students' activity were transcribed and analysed using the CHAT frame with close attention to interactions between the students in a group. Here, students in their groups were the subject of the activity with the object of engaging together with mathematics to promote their learning. Mediating artefacts included the modelling tasks with which they worked. The authors write:

> *The composition of the community (with their members' individual histories of previous and present engagement with mathematics), the rules (explicit and implicit) and the division of labour (which influences whose ideas are valuable or not) shape in unique ways the social interactions that occur in a group activity. These interactions determine the tools that are available to the group, which in turn mediate the sense making process and influence the outcome of the activity. (Hernandez-Martinez & Harth, 2015, p. 3.63)*

Analyses showed that students had difficulties with engaging in meaningful mathematical conversation and thinking within a group related to the wider social context of university mathematics teaching. It raises issues for teaching related to preparing students for the needs and expectations of group work that is designed for their deeper mathematical understandings.

3.3 Second year mathematics beyond lectures (SYMBoL) + peer support

The SYMBoL project was again funded by HE STEM. It was a curriculum development project in which 4 interns (mathematics students at the end of their second year) worked with lecturers to provide resources for students in two second year mathematics modules "vector spaces" and complex variables". The two modules were found 'hard' by students taking them, and the aim was to get students' perspectives on what might be provided to help.

Over 6 weeks in the summer, in consultation with the module lecturers, the interns worked on their contribution of resources, with, every day, a discussion over tea with as many of the mathematics staff (mathematicians and mathematics educators) as were around to discuss with them. These discussions were rich in mathematics – the interns brought examples from their preparation of resources and these were discussed with and between members of staff. As a result modifications were made to the resources and interns became more confident in sharing their activity with staff. Staff felt they learned much about student perspectives through this engagement. Thus, students and staff felt the mutual understandings that developed were important to staff-student relations in the department. In the following academic year, each module was taught using the materials the students had designed.

An important outcome from the project was the creation of a peer support system in which third year students (having been 'trained' by staff in the Mathematics Education Centre and University Teaching Centre to enact a student-centred pedagogy) held (voluntary) tutorials each week with the second year students taking the two modules,. Research demonstrated that the second year students who participated in these tutorials had a higher achievement in their final examinations, even after controlling for their lecture attendance and prior attainment (Duah, Croft & Inglis, 2013). A thesis documenting this study is forthcoming (Duah, in press).

Key learning outcomes from these studies

The three examples provided above have a number of aspects in common (beyond their common university base). All are research and development projects. The first two are overt in their use of small-group activity to promote student involvement and engagement with mathematics. Both show that pedagogic theory is not sufficient in and of itself to ensure students learning through (well designed) pedagogy. Both involve teaching mathematics to engineering students and this raises questions about the needs of engineering students in contrast with students studying mainstream mathematics. The third project was conducted with mathematics students engaging with more traditionally taught mathematics. Here, it was the students themselves who were bringing the more innovative pedagogy to the modules.

The sociocultural nature of all three projects means that analyses are conducted with attention to the wider social influences on (innovative) teaching-learning processes. They reveal characteristics of the university (mathematics) cultures which impinge on students' attitudes and approaches to their learning. So, we cannot focus only on the mathematics and on students' mathematical cognition if we want to explore and develop teaching approaches to improve student success. These studies are starting to point to issues in teaching and learning of which university teachers need to be aware, particularly since students entering university are increasingly less well prepared for university mathematics study.

Development and research

Developmental research is research which both studies the developmental process and contributes to development (Jaworski, 2003). Contribution to development might be either implicit or explicit. In Jaworski, Mali & Petropoulou (in review), we point to a number of studies of university mathematics teaching in which development was largely implicit, and (largely) not a focus of the research. In other words, as researchers we are aware of development that came along with the project, but is not documented through analyses and findings. In the first and second examples above, development was an important part of the research design since the teacher in each case was a member of the team conducting the research, and had the overt intention to learn from the research process and to develop teaching. Such learning is both dynamic in being fed back to the ongoing teaching as teaching progresses from year to year (enhancing knowledge in practice), and potentially influential in being communicated through research reporting at both professional and scientific levels (enhancing knowledge in the wider community).

In the third project, the teachers were established research mathematicians, both working in recognisably traditional lecturing modes. It is undoubted that they learned during the active search for new resources for their module, as did many of their colleagues who were a part of the tea-time discussions in the project. What is documented is the activity of the interns, the resources they designed, and perspectives of the teaching staff (Duah, in press). We are not able to report on whether or how the subsequent teaching of the two modules developed from this activity. However, as mentioned above, the peer support activity had important outcomes for students' learning of mathematics in the two modules. The student-centred pedagogy developed with the peer leaders was an important outcome of the entire project. A bi-product of this project was a positive development in relations between mathematics educators leading the project and mathematicians participating in it. Such joint projects have an important role to play in bringing these two groups closer in understanding issues in teaching and pedagogy.

In the above, I have sought to demonstrate the issues arising when mathematics education researchers study aspects of learning and teaching mathematics at university level, the learning derived from such research and the need for more of it. The contrasting of different forms of pedagogy provides insights into ways in which pedagogy relates particularly to the learning of *mathematics.* Sociocultural frames allow a relating of students' learning of mathematics and the teaching they experience with the cultural and systemic issues affecting what is possible in university teaching settings. Research in these areas is still in its infancy.

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Coordinating analyses of individual and collective mathematical progress

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A challenge in mathematics education research is to coordinate different analyses to develop a more comprehensive account of teaching and learning. We contribute to these efforts by expanding the constructs in Cobb and Yackel's (1996) interpretive framework that allow for coordinating social and individual perspectives. This expansion involves four different constructs: disciplinary practices, classroom mathematical practices, individual participation in mathematical activity, and mathematical conceptions that individuals bring to bear in their mathematical work. The first two constructs offer insights into the mathematical progress of the classroom community while the latter two address the progress of individual students. While the four analyses are informative in their own right, power is added with a discussion of combining and coordinating across the four analyses. Such networking strategies and methods have considerable potential for increasing explanatory, descriptive, and prescriptive power.

Introduction

Recent work in mathematics education research has sought to integrate different theoretical perspectives to develop a more comprehensive account of teaching and learning (Cobb, 2007; Hershkowitz, Tabach, Rasmussen, & Dreyfus, 2014; Prediger, Bikner-Ahsbahs, & Arzarello, 2008). One of the early efforts at integrating different theoretical perspectives is Cobb and Yackel's (1996) emergent perspective and accompanying interpretive framework. In this paper we expand the interpretive framework for coordinating social and individual perspectives by offering a set of four constructs for how to examine the mathematical progress of both the collective and the individual. In the full paper we use data from an undergraduate mathematics course in differential equations to illustrate these constructs by conducting four parallel analyses and make initial steps toward coordinating across the analyses. In this expanded abstract we introduce the four constructs.

In the interpretative framework, classroom mathematical practices, a collective construct, are viewed as reflectively related to the individual construct of conceptions and activity. Grounded in a need to more completely account for undergraduate students' mathematical activity and the desire to connect with existing cognitively-based literature, we expand these two constructs into the following four constructs: disciplinary practices, classroom mathematical practices, participation in mathematical activity, and mathematical conceptions. While each of the four constructs are informative in their own right, power is added when one combines and coordinates analyses across the four constructs. Indeed, Prediger, Bikner-Ahsbahs, and Arzarello (2008) argue that such networking strategies and methods are sorely needed, and they describe the benefits that such a coordination or combination affords. For instance, they state that "developing empirical studies which allow connecting

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theoretical approaches" may further the scientific discipline of mathematics education research by allowing us "to gain an increasing explanatory, descriptive, or prescriptive power" (p. 169).

The four analytic constructs and respective research questions are shown in Figure 1.

Disciplinary practices	Classroom mathematical practices	Participation in mathematical activity	Mathematical conceptions
What is the mathematical progress of the classroom community in terms of the disciplinary practices of mathematics?	What are the normative ways of reasoning that emerge in a particular classroom?	How do individual students contribute to mathematical progress that occurs across small group and whole class settings?	What conceptions do individual students bring to bear in their mathematical work?

Figure 1. Four constructs for analyzing mathematical progress and respective research questions

In the next section we operationalize each of these constructs and then conclude with a discussion on various ways for coordinating analyses.

Analytic Constructs

Classroom mathematical practices. Classroom mathematical practices refer to the normative ways of reasoning that emerge as learners solve problems, explain their thinking, represent their ideas, etc. By normative we mean that there is empirical evidence that an idea or way of reasoning functions as if it is a mathematical truth in the classroom. This means that particular ideas or ways of reasoning are functioning in classroom discourse *as if* everyone has similar understandings, even though individual differences in understanding may exist. The production of these normative ways of reasoning constitute the mathematical progress of the classroom community. The empirical evidence needed to document normative ways of reasoning is garnered using the approach developed by Rasmussen and Stephan (2008) and Stephan and Rasmussen (2002). This approach applies Toulmin's argumentation scheme to document the mathematical progress.

In his seminal work, Toulmin (1958) created a model to describe the structure and function of argumentation. The core of an argument consists of three parts: the data, the claim, and the warrant. In an argument, a speaker or speakers makes a claim and presents evidence or data to support that claim. Typically, the data consist of facts or procedures that lead to the conclusion that is made. To further improve the strength of the argument, speakers often provide more clarification that connects the data to the claim, which serves as a warrant, or a connector between the two. Finally, the argumentation may also include a backing, which demonstrates why the warrant has authority to support the data-claim pair. Toulmin's model also includes qualifiers and rebuttals. To document normative ways of reasoning, one begins by using Toulmin's model to code every whole class discussion, resulting in anywhere from a few to more than a dozen coded arguments. The collection of all coded arguments results in an argumentation log for all whole class discussions. The next step involves taking the argumentation log as data itself and looking across all class sessions to see what mathematical ideas become part of the class' normative ways of reasoning. The following two criteria are used to determine when a way of reasoning becomes normative:

Criterion 1: When the backings and/or warrants for a particular claim are initially present but then drop off. For example, criterion 1 is satisfied when the same claim gets debated on more than one class period or more than once during the same class period and in subsequent occurrences the backing or warrants drop off.

Criterion 2: When certain parts of an argument (the warrant, claim, or backing) shifts position within subsequent arguments, indicating knowledge consolidation. For example, criterion 2 is satisfied when once-debated conclusions shift function over time and serve as unchallenged data or justification for future conclusions.

The use of this methodology requires classrooms in which genuine argumentation is a norm. That is, students are routinely explaining their reasoning, indicating agreement or disagreement with other's reasoning, etc.

Disciplinary practices. Disciplinary practices refer to the ways in which mathematicians go about their profession. The following disciplinary practices are among those core to the activity of professional mathematicians: defining, algorithmatizing, symbolizing, modeling, and theoremizing (Rasmussen, Zandieh, King, & Teppo, 2005). Not all classroom mathematical practices are easily or sensibly characterized in terms of a disciplinary practice. This is because classroom mathematical practices capture the emergent and potentially idiosyncratic collective mathematical progress, whereas a disciplinary practice analysis seeks to analyze collective progress as reflecting and embodying core disciplinary practices. For example, an important algorithm in differential equations is Euler's method, which is a numerical technique for obtaining an approximate solution to an initial value problem. When students have opportunities to create and use an algorithm, such as Euler's method, they are positioned to participate in the disciplinary practice of algorithmatizing. The term algorithmatizing is similar to the term "theoremizing" in the following way. Each has a noun as the root (algorithm and theorem) made into a verb. The verb form reflects a focus on student activities, namely creating and using algorithms in the former and conjecturing and proving in the latter. When students are engaged in genuine argumentation it is often the case that conjectures are made and then justifications are created to support or refute the conjectures. The term theoremizing is used to explicitly encompass both conjecturing and steps toward justifying the assertions.

Our use of the term "disciplinary practice" is somewhat similar to how Moschkovich (2007) describes "professional discourse practices", which includes the discourse practices of academic mathematicians. We agree with Moschkovich that such practices are culturally and historically situated. Moreover, while perhaps not all academic mathematicians would characterize their work in terms of defining, algorithmatizing, symbolizing, and theoremizing, we argue that these broad categories do capture much of what professional mathematicians do and represent what Moschkovich (2007) argues are "socially, culturally, and historically produced practices that have become normative" (p. 25). In our analysis of classroom data, however, we employ a grounded approach (Glaser & Strauss, 1967) to characterize the ways in which the students engage in these broader disciplinary practices. That is, we do not impose any set of a priori categories of student activity related to defining, algorithmatizing, symbolizing, or theoremizing, but rather allow the data to shape how we characterize the features of a disciplinary practice that emerge in a particular class.

Mathematical conceptions. As students solve problems, explain their thinking, represent their ideas, and make sense of others' ideas, they necessarily bring forth various conceptions of the ideas being discussed and potentially modify their conceptions. From this point of view, we seek to answer the question: What conceptions do individual students bring to bear in their mathematical work? For example, in the inquiry oriented differential equations class where students reinvented Euler's method, individual students thought about rate of change in various ways, many of which are exemplified in the literature on ratio and rate (e.g., Thompson, 1994; Zandieh 2000).

Analyses of individual student conceptions can make use of constructs from prior work that have characterized different views that students can have of key mathematical ideas. Indeed, there is a rich literature that has characterized various ways that students might think about particular ideas in linear algebra, analysis, differential equations, and abstract algebra. For example, both Sierpinska (2000) and Hillel (2000) developed overarching frameworks for analyzing student reasoning across the linear algebra curriculum. Other studies analyzed student difficulties with the notions of basis, linear transformation, and rank, among other concepts.

Participation in mathematical activity. This construct for analyzing individual mathematical progress is used to answer the question: How do individual students contribute to the mathematical progress that occurs across small group and whole class settings? To address this question, our approach draws on recent work by Krummheuer (2007, 2011). Krummheuer characterizes individual learning as participation within a mathematics classroom using the constructs of production design and recipient design. In production design, individual speakers take on various roles, which are dependent on the originality of the content and form of the utterance. The title of *author* is given when a speaker is responsible for both the content and formulation of an utterance. The title of *relayer* is assigned when a speaker is not responsible for the originality of either the content nor the formulation of an utterance (i.e., responsible for neither content nor form). A *ghostee* takes part of the content of a previous utterance and attempts to express a new idea (i.e., is responsible for content but not form), and a *spokesman* is one who attempts to express the content of a previous utterance in his/her own words (i.e., is responsible for form but not content).

Within the recipient design of learning-as-participation, Krummheuer (2011) defines four roles: conversation partner, co-hearer, over-hearer, and eavesdropper. A *conversation partner* is the listener to whom the speaker seems to allocate the subsequent talking turn. Thus, the conversation partner is not only directly addressed but also evidences a high level of engagement. Listeners who are also directly addressed but do not seem to be treated as the next speaker are called *co-hearers*. Whereas the previous two listening roles involved direct participation of the recipient to the utterance, the final two involve indirect participation. Those who seem tolerated by the speaker but do not participate in the conversation are *over-hearers*, and listeners deliberately excluded by the speaker from conversation are *eavesdroppers.*

Conclusion

At a minimum, the four constructs provide an opportunity to analyze the same phenomenon from four distinct points of view – as if one were gazing at the same object from various vantage points in order to capture many qualitative nuances about the object. In addition to using various combinations of the four constructs to more fully interpret students' mathematical progress, there exist multiple ways in which coordination across the four constructs is possible. For instance, one could choose an individual student within the classroom community and trace his/her utterances for the ways in which they contributed to the emergence of various normative ways of reasoning and/or disciplinary practices. Alternatively, when considering a normative way of reasoning, a researcher could investigate who the various individual students are that are offering the claims, data, warrants, and backing in the Toulmin schemes used to document the normative way of reasoning. How do those contributions coordinate with those students' production design roles within the individual participation construct? For instance, does a student ever utilize an utterance that a different student authored as data for a new claim that he is authoring, and in what ways may that capture or be distinct from other students' individual mathematical conceptions? We also imagine ways to coordinate across the two individual constructs as well as across the two collective constructs. For example, how do patterns over time in how student participation in class sessions relate to growth in their mathematical conceptions? Are different participation patterns correlated with different mathematical growth trajectories? In what ways are particular classroom mathematical practices consistent (or even inconsistent) with various disciplinary practices? Finally, research could take up more directly the role of the teacher in relation to the four constructs.

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1. MATHEMATICS AS A SUBJECT IN PRE-SERVICE TEACHER EDUCATION

Transforming aspirations of future mathematics teachers into strategies in context

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In this paper I present a collaborative research and development programme, in which we design situation specific tasks and use them to explore, challenge and change knowledge and beliefs of in- and pre-service secondary mathematics teachers. In this work we use practice-based and research-informed tasks in which we invite teachers to consider a mathematical problem and typical student responses (and teacher reactions) to this problem. So far the programme develops in four stands: (1) mathematical knowledge for teaching; (2) classroom management and mathematics learning; (3) disability and inclusion in the mathematics classroom; and, (4) meta-use of tasks and task development. Examples of tasks from these strands will be discussed in the session.

Introduction

In this paper I present a collaborative research and development programme since 2005 on secondary mathematics teachers' knowledge and beliefs and the transformation of these knowledge and beliefs into pedagogical practices. Research acknowledges the overt discrepancy between theoretically and out-of context expressed teacher beliefs about mathematics and pedagogy and actual practice (e.g. Speer, 2005) and a substantial body of work in mathematics education explores the use of specific teaching cases (e.g. Markovits and Smith, 2008) in teacher education. Our research sets out from the assumption that teacher knowledge is better explored and developed in situation-specific contexts and to this aim we design *situation specific tasks (thereafter Tasks)* – i.e. tasks based on specific mathematical teaching scenarios – and then use them for research and teaching purposes. These classroom scenarios: are hypothetical but grounded on learning and teaching issues that previous research and experience have highlighted as seminal; are likely to occur in actual practice; have purpose and utility; and, can be used both in (pre- and in-service) teacher education and research through generating access to teachers' views and intended practices.

So far, seven mathematics educators from the UK, Greece and Brazil have been involved in this programme and the research we have conducted – and we anticipate to conduct in the following years – is divided in four strands: (1) *mathematical knowledge for teaching* (e.g. mathematical thinking; pedagogical and didactical practices in the mathematics classroom); (2) *classroom management and mathematics learning* (e.g. interference of the classroom management with the learning of mathematics); (3) *disability and inclusion in the mathematics classroom* (e.g. deaf and blind students strategies in dealing with mathematical problems); and (4) *meta-use of tasks and task development* (e.g. asking teachers to create their own classroom situations and tracking the impact this engagement has on their knowledge and beliefs). The format of the *Task* varies across the programme – e.g., monologue or dia-

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logue; script or video clip format; one or more students; teacher intervention or not; etc. – in order to address different issues and different aspects of these issues in relation to the teaching and learning of mathematics. In the following sections I describe briefly each one of these strands.

Mathematical knowledge for teaching

In the *Tasks* of this strand we invite teachers to: solve a mathematical problem; examine a (fictional yet research-informed) solution proposed by a student (or more than one students) in class and, in some versions, a (fictional yet research-informed) teacher response to the student; and, describe the approach they themselves would adopt in this classroom situation (Biza, Nardi & Zachariades, 2007, 2009; Nardi, Biza & Zachariades, 2012; Zachariades, Nardi & Biza, 2013).

From the teachers' responses to these *Tasks* we aimed to explore teachers' subject matter knowledge and their gravitation towards certain types of pedagogy and didactical practices (Biza *et al.*, 2007). So far, teacher responses in these *Tasks*, joined with *post-Task individual semi-structured interviews*, have allowed us to access a range of teacher knowledge and beliefs (epistemological and pedagogical). For example, in (Biza *et al.*, 2009) we discuss the multiple didactical contracts on the role of visualisation in mathematics and mathematical learning that teachers are likely to offer their students under those influences (e.g. is a graph-based argument an acceptable argument in the mathematics classroom?). Additionally, teachers' responses to these tasks and interviews with them revealed a complex set of considerations that teachers take into account when they determine their actions (Nardi *et al.,* 2012) – what Herbst and colleagues (e.g. Herbst and Chazan 2003) describe as the *practical rationality of teaching.* We demonstrate how teacher arguments, not analysed for their mathematical accuracy only, can be reconsidered, arguably more productively, in the light of other teacher considerations and priorities: pedagogical, curricular, professional and personal that influence the decisions teachers make in the classroom. Recently, we introduced a format in which an elaborated design that enriches and develops the previous one in which apart from the student flawed (fictional) response(s), a fictional response from a teacher has been added (Zachariades, *et al.* 2013). With this design we aim to explore, not only whether the teacher can identify a student mathematical error and what their pedagogical intentions are, but, also, how they evaluate the pedagogical approach followed by another (fictional) teacher.

Classroom management and mathematics learning

The motivation for this strand came from the research and practice based observation that classroom management often interferes with working towards commendable learning goals (e.g. Kersting, 2008). The *Tasks* we designed for this strand are based on realistic classroom scenarios that combine seminal mathematics learning and teaching issues with classroom behaviour issues (e.g. classroom management, conflicts between students or between students and teacher). For example, in one of these *Tasks* a class is asked to solve the problem: "When p = 2.8 and c = 1.2, calculate the expression: $3c^2 + 5p - 3c(c-2) - 4p$". Two students reach the result (10) in different ways: Student A substitutes the values for p and c and carries out the calculation; Student B simplifies the expression first and then substitutes the values for p and c. When Student A acknowledges her difficulty with simplifying expressions, Student B retorts offensively ("you are thick") and dismissively ("what can I expect from you anyway?"). Both solutions are correct and Student B's approach particularly demonstrates proficiency in important algebraic skills. But Student B's behaviour is questionable. 21 prospective mathematics teachers were asked to write, and then discuss, how they would handle this classroom situation. Results indicate commendable norms teachers aspire to establish in their classroom: peer respect; value of discussion; and, investigative mathematical learning. However, they often miss the opportunity to engage students with metacognitive discussions and mathematical challenge as they focus on behavioural issues or endorse dichotomous and simplistic views of mathematical learning (Biza, Nardi & Joel, 2015).

Disability and inclusion in the mathematics classroom – CAPTeaM

This is a recent development of our programme and relates to inclusive education and teacher perspectives on how students with disabilities (in our studies so far deaf, blind and with Down syndrome) learn mathematics. Our project is called CAPTeaM (Changing Ableist Perspectives on the Teaching of Mathematics) and is funded by the British Academy. Specifically, according to the ableist world-view, the able-bodied are the norm in society and disability is an unfortunate failing, a disadvantage that must be overcome. Within education, ableism results in institutional and personal prejudice against learners with disabilities, and has a drastic effect on approaches to teaching (Nardi, Healy & Biza, 2015). Our project investigates how ableist perspectives impact on the teaching of mathematics, a discipline where public perceptions of ability as innate often shape pedagogical perspectives and practice. In this strand the expertise of a team of Brazilian researchers (Lulu Healy and colleagues) on mathematics learners with disabilities joined with our *Task* design approaches to develop and trial *Tasks* that invite teachers to reflect upon the challenges of mathematics teaching in inclusive classrooms. The *Tasks* in this strand are of two types. In Type I, using the approach described by Biza *et al.* (2007), the scenario is inserted as a video clip into a brief narrative about a fictional mathematics classroom. We then invite participants (prospective mathematics teachers) to assume the role of the teacher of this class and evaluate the interactions of the disabled students that were presented in the video clips – first individually and in written responses to a set of questions, and then in a group discussion (which we also video-record). In the tasks of Type II, which aim to provoke reflections about how access to mediational means differently shapes mathematical activity, participants work in groups of three. Two members of the group are asked to solve a mathematical problem whilst, temporarily and artificially, deprived of one of their sensory or communication canals. A group discussion of their experiences follows.

For example, in one of the Type I *Tasks* students work on exploring how they would describe what a square-based pyramid is to someone who doesn't know. André, who is blind, and has been working with 3D solids, offers a description (seen by participants in a video clip) shaped around the idea of a square based pyramid being built out of gradually shrinking squares. Preliminary analysis of 81 responses indicates, for example, preference for switching André's perspective on a square-based pyramid towards the textbook definition of a pyramid (faces, edges and vertices) and preference for a discussion of a square-based pyramid as a composition of fixed shapes (four triangles and a square) (Nardi *et al.*, 2015).

Teachers' Narratives: Meta-use of tasks and task development

This strand is in resonance with works such as Zazkis, Sinclair and Liljedahl (2013) on teachers' creation of their own lesson plays. We started working on this direction last year when we invited prospective teachers to write a brief teaching/learning scenarios from their own first experiences from schools. We collected 12 scenarios, we grouped them thematically and we invited trainees to discuss these in groups, produce posters of the key points of the discussion and then share these points with the whole group. The themes we identified concern issues such as mathematical learning (e.g. misconceptions, instrumental and relational understanding); classroom management; student engagement; and, prospective teachers' relationships with more experienced teachers. Group and class discussion were audio-recorded and transcribed. We are now analysing these in close collaboration with practising teachers. We see these narratives as opportunities for teachers' reflection on their practice. Furthermore, we see the benefits of the collaboration of researchers and teachers in the analysis of these narratives in both research and professional development.

Conclusion

In conclusion, from our research the last 10 years, we credit this task design with allowing insight into pre- and in-service teachers' considerations. Teachers very often express commendable aspirations without, however explaining how they would transform these aspirations into practice. We propose the further implementation of this situation specific task design in teacher education programmes towards the transformation of these aspirations of future mathematics teachers into teaching strategies.

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How can primary teacher education students' achievement in geometry be improved? Results from the KLIMAGS project

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The KLIMAGS Project aimed at knowing more about the knowledge, competencies, beliefs, interest, and strategies that beginning primary teacher education students have in arithmetic and geometry, how that knowledge etc. develops in the first year of university studies, and what effects targeted innovations in university courses in arithmetic and geometry have on this development. The presentation focuses on the research design of KLIMAGS and on results concerning courses in geometry at the University of Kassel.

Introduction

In the last fifteen years, there have been several studies that have investigated the professional competence of future or present mathematics teachers. The first such study was the Michigan project (see Ball & Bass, 2003 or Hill, Rowan & Ball, 2005) that conceptualized and measured the „mathematical knowledge for teaching" of primary school teachers. They distinguished, following a well-known categorisation suggested by Shulman (1986), among content knowledge (CK) and pedagogical content knowledge (PCK), and further among „common", „specialized" and „horizon" content knowledge respectively among knowledge of „content and students", „content and teaching" and „content and curriculum". The MT21 and TEDS-M projects (see Schmidt, Tatto, Bankov et al., 2007, and Tatto, Schwille, Senk et al., 2008; in particular for the results of the German project component see Blömeke, Kaiser & Lehmann, 2008, and Blömeke, Kaiser & Lehmann, 2010) investigated, comparatively on an international level, the mathematical CK and PCK of future primary and secondary school teachers. They found considerable differences in students' knowledge, strongly correlated to the learning opportunities in teacher education. The COACTIV project (see Kunter, Baumert, Blum et al., 2013) studied the CK and PCK of a representative sample of German secondary school teachers. Because the study was linked to the longitudinal component of the German PISA study 2003/04, COACTIV could link the teacher data to the student data and detected strong correlations between teachers' CK and PCK, on the one hand, and between PCK, aspects of instructional quality and students' learning progress on the other hand (see Baumert, Kunter, Blum et al., 2010, for details).

So in all studies, the future or practising teachers' CK proves to be a highly important component of teachers' professional competence. It is a major task of pre-service teacher education to supply future teachers with the necessary CK. At the same time, the TEDS study has revealed considerable shortcomings in German primary school teachers' CK (see Döhrmann, 2012, for details). This finding is in accordance with the unsatisfactory results of primary school students' performance in various university examinations that most university teachers experience regularly. That was the starting point of the *KLIMAGS project*.

The KLIMAGS Project

The *KLIMAGS Project*, embedded in the khdm centre for university mathematics education research (www.khdm.de), investigates the *mathematics courses for first year primary school students* at the universities of Kassel and Paderborn. KLIMAGS started in October 2010, directed by P. Bender, R. Biehler, W. Blum and R. Hochmuth, and aimed at knowing more about the knowledge, competencies, beliefs, interest, and strategies which beginning primary education students have in arithmetic and geometry, how that knowledge etc. develops in the first year of university studies, and what effects targeted innovations in university courses in arithmetic and geometry have on this development. The research design of KLIMAGS was as follows. The student cohort 2011/12 was the control group (CG), both in Kassel and in Paderborn, with courses in arithmetic and geometry as taught in the years before; there were four points of measurement in the first two semesters. The student cohort 2012/13 was the experimental group (EG), with certain innovations in these courses, and the same points of measurement. For these measurements, special achievement tests were developed, with 52 items in arithmetic and 26 items in geometry. All tests were IRT scaled, with EAP/PV reliabilities between .75 and .85 and item parameters ranging from –3.3 to 3.6.

The KLIMAGS sub-study "Geometry Kassel"

Our presentation in Hannover will concentrate on the courses in *geometry* in *Kassel*. In summer semester 2012, the beginners (CG) were investigated in a pre-/post-test design, and in summer semester 2013 the corresponding cohort (EG) in the same way. The *innovation* in the EG lecture and the accompanying written exercises consisted of a treatment of all modes of representation (enactive, iconic, symbolic) for the topic of congruence mappings, an explicit change between these representations as well as a meta-cognitive explication of connections and a reflection on the relevance for students' learning, whereas in the CG lecture only iconic and symbolic representations were treated, with deliberately no meta-cognitive elements. The rationale for this innovation is rooted in well-known findings about positive effects of a transfer between different modes of representation (mainly on the school level). The innovation in the EG tutorials consisted of a professionalization of the tutors of the course, in particular through tutor training in diagnosis, feedback and learning support; in parts we could draw on experiences from the LIMA project (see Biehler, Hänze, Hochmuth, Becher, Fischer, Püschl et al., 2013). This innovation is rooted in recent findings about the importance of the professional knowledge of teachers for instruction quality as well as for the learning progress of their students (see introduction). All other aspects of the two courses (especially the lecturer) were as identical as possible, both in content and in method. The time which was used in the EG for the transfer between representations and for discussions on the meta-level was used in the CG for further examples of concrete congruence mappings. Treatment control took place by analyses of lecture scripts and students' written exercises. Our *research questions* were whether this intervention actually results first in a better understanding of EG students compared to CG students for the content area of congruence mappings and second in a significantly higher achievement progress of the EG, mainly resulting from their higher progress in the innovated content area.

Results

Our *sample* consisted of 255 students, mainly in their first year of study. The sub-sample relevant for the evaluation consisted of the 98 students who have taken part in both pre- and post-tests, 53 in the CG and 45 in the EG. On a 5 % level, there was no significant difference in the test results between the total sample and the sub-sample of 98 students. We had a rotation *test design* with two test versions, 13 of the 26 items identical and 13 items rotating.

The main *quantitative results* are given in the following table (obtained by variance analyses with repeated measurements over latent person abilities).

	Pre-test geometry	**Post-test geometry**
CG	**-1.01 (0.86)**	**0.36 (0.84)**
EG	**-1.18 (0.96)**	**0.66 (1.14)**

Thus, the essential results are:

- The pre-test results of CG und EG are not significantly different (t-test, $t(96) = 0.896$, $p = .372$)
- Both groups show a big achievement progress (so both courses were efficient)
- The achievement progress of the EG is significantly higher (ANOVA and F-test, $F(1) = 4.766$; $p < .05$; $\eta^2 = .047$)

So, the results show the expected advantage of the EG, and a closer inspection shows strong effects of items from the innovated content area (the number of items is too small for a split into two sub-tests, like we did in arithmetic). With our design, we cannot disentangle the effects of the innovation in the lecture and of the innovation in the tutorials. The test results indicate that the main source for the effects is the innovation in the lecture, perhaps moderated by the innovation in the tutorials.

The advantage of the EG is also revealed by *qualitative analyses* of students' solutions. For instance, for an item where the students had to compose two symmetry mappings (a reflection and a rotation) of an equilateral triangle, the solution rates in the post-test were 37 % in the CG and 51 % in the EG. Of course, such results are still disillusioning from a normative point of view.

Outlook

For the courses in *arithmetic* in Kassel we have obtained similar results. The intervention was analogous: multiple representations, change between them and meta-cognitive explication for the content area of divisibility rules based on position systems. Here, we found significant advantages for the EG in the sub-test corresponding to the innovation and no differences in the rest of the test (see Blum, Biehler, Hochmuth, Bender, Kolter, Haase et al.

in preparation). So an obvious conclusion from both courses is that lecture innovations as implemented in Kassel seem promising. However, the aim ought to be to have much more transfer to other content areas, what perhaps can be reached by even more meta-cognitive explications and reflections and a stronger connection of the mathematics courses with the corresponding courses in didactics of mathematics. And there have, of course, to be reinforced efforts to raise primary students' interest in mathematics, to change their beliefs, to advance strategies, and thus to contribute to a further improvement of their achievement progress.

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Disagreements between mathematics at university level and school mathematics in secondary teacher education

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In this paper, a project regarding developmental aspects as well as research aspects will be discussed. The project essentially concerns prospective mathematics teachers and their perceived disagreements between university mathematics and school mathematics that Felix Klein called double discontinuity. Firstly a motivation for the mentioned project is given. Afterwards the current schooling of prospective mathematics teachers is sketched and ways of expanding the traditional schooling in terms of reducing the perception of a double discontinuity are outlined with a few examples. Furthermore, first findings from quantitative and qualitative pilot studies regarding the beliefs of prospective teachers are provided.

Background

At the beginning of the last century, Felix Klein mentioned in the preface of one of his textbooks the notion of a "double discontinuity" in the mathematical socialization of mathematics teachers (Klein, 1908, 1). This term characterizes the awareness of discrepancies between school mathematics and academic mathematics that prospective teachers have to deal with, and also the transition from university studies to a professional career as a mathematics teacher in school. As a consequence of the perception of a double discontinuity, prospective teachers may lose sight of university mathematics after their exams and, thus, teach on the basis of experiences from their own schooldays (cf. Hefendehl-Hebeker, 2013). Even nowadays, this phenomenon still seems to exist, and prospective teachers frequently believe that the topics of university mathematics do not meet the demands of their later profession in school (ibid.).

Considering this background, there is a need to reduce the discontinuities between university mathematics and school mathematics. Taking the experience of scholars into account, prospective teachers are mostly not able to make the connections between school mathematics and university mathematics on their own. Also, remarks by lecturers in this regard do not seem to have sufficient results (cf. Bauer, 2013). Developing the desired bridges and establishing such bridges in secondary teacher education, particularly in the first two years of the university studies, is the main aim of a project at the University Kassel called f-f-u (integration of mathematics and mathematics education at university) that combines development and research. The mentioned concept is part of a larger project named PRONET (professionalization by integration) designed to promote teacher education in different faculties. In this paper, we briefly discuss how an integration of university mathematics and aspects of teaching mathematics could be applied to reduce the mentioned double discontinuity in the mind of prospective teachers.

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University mathematics for prospective teachers

Upper secondary teachers in Germany are obligated to take two basic subjects during their university studies, e.g. mathematics and chemistry, and also to enroll some general courses in pedagogy, psychology, etc. Therefore (and in contrast to other countries) prospective teachers have got a smaller amount of activities in their basic subjects in comparison to students who follow a major programme in these subjects. However, as usual at German universities, prospective mathematics teachers at the University Kassel are enrolled in the same mathematics courses as mathematics majors, particularly in the first semesters. The mathematics courses, e.g. analysis, usually include four hours per week of plenary lectures plus two additional courses in a week in which student teachers organize exercises in small groups. These exercises rely on homework of the university students and contain a range of tasks that can be solved on the basis of the plenary lectures. The prospective teachers' performance referring homework yields to an admission for final exams referring to the course (e.g. analysis).

Within the scope of homework and exercises the mentioned integration of mathematics and mathematics education is intended. The main idea of the project f-f-u is to enrich the set of tasks for homework with *teacher-oriented tasks* that are appropriate to illustrate connections between university mathematics and school mathematics. Aspects of developing these tasks as well as some examples are outlined in the following paragraph.

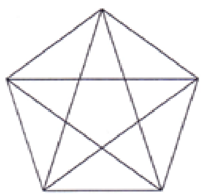
Examples

Actually, there are two directions of emphasizing bridges between school mathematics and university mathematics (Bauer, 2013).

The first direction refers to bridging school mathematics and university mathematics. Specific tasks that we call *teacher-oriented tasks* could for example illustrate a need for advanced mathematics when dealing with mathematical issues in school. The following examples represent appropriate questions within the scope of school mathematics.

Look at the below-mentioned task from the textbook "Bildungsstandards Mathematik: konkret"

In the regular pentagon all 5 diagonals were plotted.



Complete all numbers of diagonals of the given regular polygons in the following scheme.

Triangle	*tetragon*	*pentagon*	*hexagon*	*heptagon*	*octagon*	*dodecagon*
		5				

Describe how you found the number of diagonals in the dodecagon.

1) Solve this task as you would expect it from a student in grade nine.

2) Find a general formula for the number of diagonals in a regular polygon with n angles. Can you also show proof to your statement?

Fig. 1: Task "diagonals" for prospective mathematics teachers (cf. Blum et al., 2006, 121)

We want to add the areas of two rectangles. However, the edges of these rectangles overlap. Is the consequence of doing so that one of the tetragons has a smaller area?

Fig. 2: Task "rectangles" for prospective mathematics teachers

The second direction of emphasizing bridges between school mathematics and university mathematics includes the awareness that school-related questions sometimes need a deeper investigation represented by university mathematics. For example that means to analyze mathematical problems in school from a higher standpoint. Referring to the distinction of teachers' professional knowledge by Ball et al. (2008), these tasks may include aspects of specialized content knowledge (SCK) as well as teachers' knowledge of content and teaching (KCT) and also teachers' knowledge of content and students (KCS).

In your lesson a discussion started about the question, whether $0.\overline{9}$ is equal to 1 or not.

One student supposes: "Already in appearance, the figure $0.\overline{9}$ looks smaller than 1." Another student states: "I guess $0.\overline{9}$ is equal to 1, but then it is somewhat rounded off." Someone else considers: "One cannot decide this, because the infinite queue is incredible."

1) How would you comment on this topic? Can you give an adequate and school-oriented explanation to your students?

2) Provide a mathematical substantiation by means of the lecture about infinite series.

Fig. 3: Task "period" for prospective mathematics teachers

In the lecture we examined the derivative of functions of multiple variables.

Now consider of introducing the derivative in school: What are the assets and drawbacks to introduce the derivative by the limit of the difference quotient or by linear approximation?

Fig. 4: Task "derivative" for prospective teachers (cf. Danckwerts & Vogel, 2006)

A task that potentially demonstrates to prospective teachers a connection between university mathematics and their own KCT could also refer to an overarching strategy of teaching mathematics, e.g. visualization (Arcavi, 2003).

In schools, the conception of continuity is often based on a visual perception. Thus, a function would be defined as continuous if the graph can be traced "without taking off the pencil".

1) Analyze the function $f(x) = \begin{cases} x \cdot \sin\frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$ in terms of continuity. Can you trace the graph of this function without taking off the pencil?

2) Discuss the potentials and obstacles of visualization in the mathematics lesson particularly when dealing with the construct of continuity.

Fig. 5: Task "continuity" for prospective mathematics teachers

Further, a teacher-oriented task could include a student's answer to an exercise that requires a deep understanding of mathematics (Tall, 1992). For example, a typical exercise in school as well as in courses of mathematics education consists of a graphical differentiation or rather integration. Therefore, a student answer of a related exercise could be a starting point in order to reconsider rules and theorems of analysis.

The derivative of a function in $x = 0$ *is* 0 $(f'(x) = 0)$. *On the left side of* $x = 0$ *the function has a positive slope, on the right side the function has a negative slope.*

Which shape of an extremum has got the function? Make a drawing of the function and give a reason for your answer.

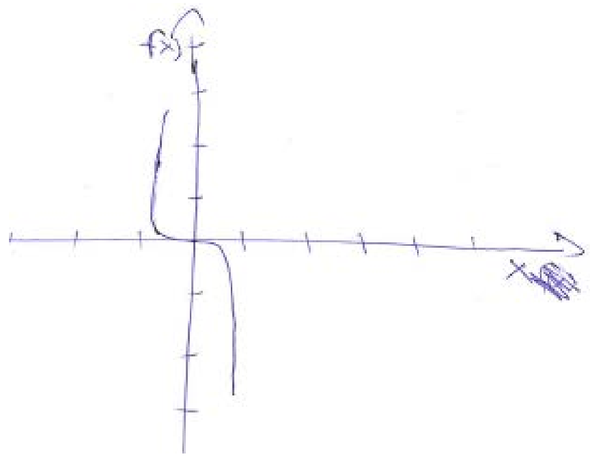


Fig. 6: A task for pupils and a student's answer

Related research

The main aim of the project f-f-u at the University Kassel is to develop an approach to integrate aspects of mathematics teaching into mathematics courses. Thus, a first research question would be how prospective teachers deal with tasks which could illustrate connections between university mathematics and school mathematics. However, the main target of the project is to investigate changes of prospective teachers' knowledge and beliefs referring to the double discontinuity that could be explained by the mentioned development of approach of courses for prospective teachers. For example, self-concept or anxiety with regard to mathematics (Hannula, 2012; Philipp, 2007) are related aspects that could be incorporated in the research by a mixed methods design (Creswell & Plano Clark, 2007) on the basis of the f-f-u program. A scale for measuring the prospective teachers' perception of a disagreement between mathematics at university level and school mathematics was piloted in the winter semester 2015/16 and is provided in the following paragraph.

In order to gain empirical evidence for the efficiency of the aforementioned method to show prospective teachers connections between university mathematics and school mathematics, students in the relevant mathematics courses will be assigned by random to a treatment group and a control group. While the control group should be taught traditionally, the treatment group will get weekly teacher-oriented homework tasks that focus on bridging mathematics at university level and school mathematics in secondary teacher education. In this paper, we refer only to pilot studies.

Pilot Studies

In the winter semester 2015/16 two basic mathematics courses at the University Kassel were selected in which the mentioned integration of mathematics and mathematics education was piloted, i.e. “principles of mathematics” and “analysis”. Prospective mathematics teachers attend these courses usually in the first or rather in the third year of their university studies. In contrast to students who major in mathematics, the teacher students got homework including an exclusive teacher-oriented task aiming to illustrate connections between university mathematics and school mathematics.

The following teacher-oriented task was inserted into the course “principles of mathematics” and concerns different forms of proof and proving (Dreyfus et al., 2012).

In the lecture we stated the theorem

For each natural number n it counts: $1 + 2 + \ldots + n = \frac{n \cdot (n+1)}{2}$

and proved it already by induction.

Now reveal this theorem for students (in grade five) in a concrete way. You can use for example figurate numbers to make this theorem plausible.

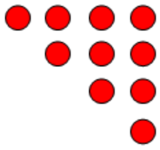
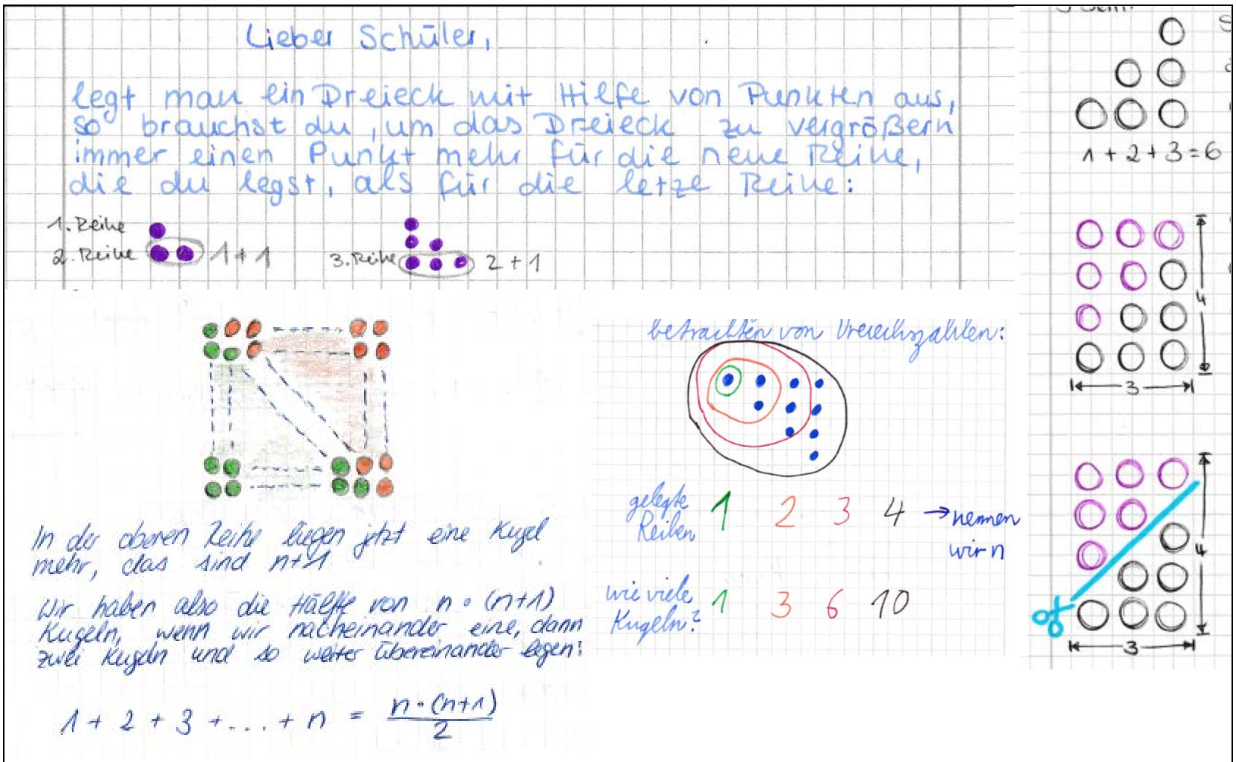


Fig. 7: Task “induction” for prospective mathematics teachers

The answers of the prospective teachers on the mentioned task were mostly elaborated and sometimes individual, too (“Dear student …”). All explanations proved to be age-based, partly even without variables or formulas. Thus, the emphasis was put on a visual approach based on generic proofs.



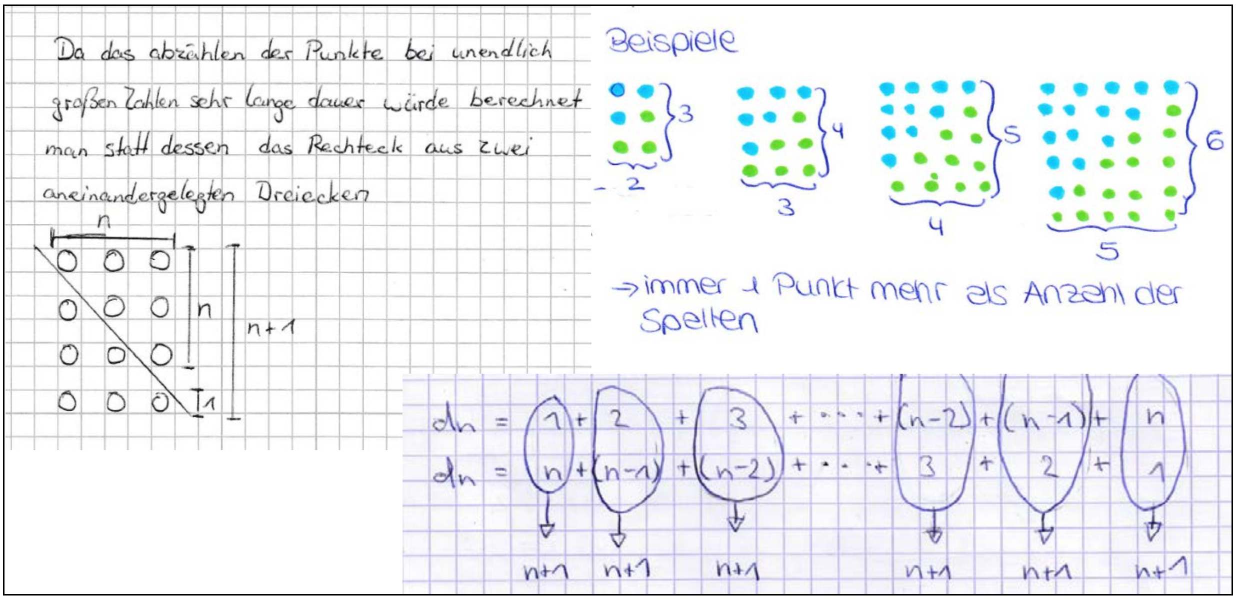


Fig. 8: Excerpts of prospective teacher's answers on the task "induction"

A further task which could lead to the experience that mathematics at university level is relevant for school mathematics was inserted into the course "analysis". This teacher-oriented task deals with a derivation rule which plays an important role in mathematics at school due to curve sketching.

The following derivation rule is a special case of the product rule as seen in the lecture:

If the function $u(x)$ is differentiable in place a, the function $f(x) = k \cdot u(x)$ (with a real factor k) is also differentiable in place a by $f'(a) = k \cdot u'(a)$

1) Give a formal proof of this derivation rule.

2) Illustrate the mentioned rule to students. You can start therefor with a polynomial of grade 2 and consider how the graph of the function $f(x)$ arises from the graph of the function $u(x)$.

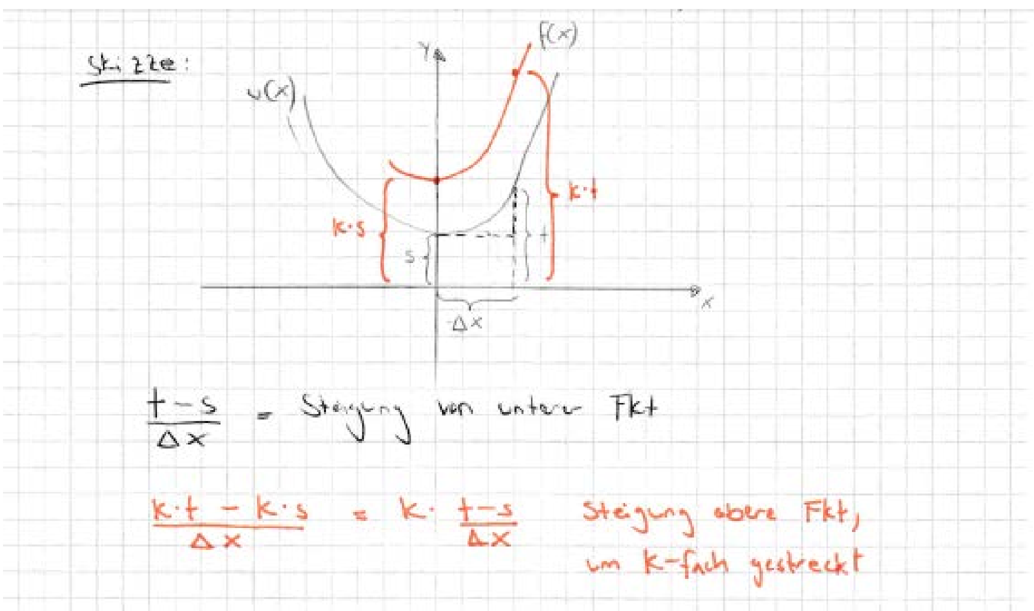


Fig. 9: Task "derivation rule" and an answer from a prospective teacher

At the very beginning of the piloting, a group of 16 prospective teachers attending the mathematics course "analysis" was asked how they managed to deal with the teacher-oriented tasks in their homework since it was the first time in their university studies that such tasks were deployed. The responses were quite different and contained comprised amazement as well as slight uncertainty:

> *„Well, up to now, I have seen from mathematics lectures that the tasks were always somehow difficult and dealing with proofs and I got adjusted to it (laughing) – but then this specific task appeared"*

> *„I noticed that this task was different and I did not exactly know, what was the expectation concerning a solution"*

> *„I solved the first part of the exercise and then thought: Where is the rub? – I must have missed something"*

Furthermore, a scale for measuring the prospective teachers' perception of a disagreement between mathematics at university level and school mathematics was piloted in a mathematics course (N = 60). All measures were taken on 6-point Likert scales[1].

University mathematics helps me to get better into school mathematics.
University mathematics does not meet the demands of my later profession in school.
I think that I require a deep understanding of mathematics in order to teach math in school.
For me, it is meaningful to deal with mathematical topics which exceed school math.
It irritates me that I have to attend mathematics courses at university.
University mathematics has mostly little relation to school mathematics.
I see correlations within school math much better by means of the mathematics course.
The mathematics course promotes me to be in thinking "one step ahead" of the students.
The relevance of university mathematics for the activity as a teacher in school is …

Tab. 1: Piloted Items concerning the perception of a double discontinuity

At a value of Cronbach's alpha 0.782 these items seem to have the potential to provide good internal consistency. Interestingly, a better result is achieved when regarding only the perspective teachers in the course (N = 35) and not all mathematics students (Cronbach's alpha 0.831).

Concluding remarks

The main topic of this paper was to discuss types of teacher-oriented tasks aiming to reduce the double discontinuity that prospective teachers might perceive when regarding the connection of university mathematics and school mathematics. We made a distinction among tasks that show connections of topics in mathematics courses at university to similar contexts in school mathematics, that show the need of sophisticated mathematics when dealing

[1] Whereas in the first eight items the university students may choose an option in a scale from "strongly disagree" to "strongly agree", the last item refers to a scale from "very low" to "very high".

with school-related mathematical problems and that show the benefit of a deep mathematical background when evaluating students' solutions. To measure changes in prospective teachers' beliefs referring to the double discontinuity, we developed a questionnaire including 9 items that actually seems to measure these beliefs.

In the following steps of our research, we will compare two groups of prospective teachers – one group in a traditional course, one group in a course using the mentioned teacher-oriented tasks – using amongst others the scale referring to the double discontinuity to prove if the type of the course has an effect of the prospective teachers' beliefs.

Acknowledgement

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Design research on inquiry-based multivariable calculus: focusing on students' argumentation and instructional design

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In this study, researchers designed and implemented an inquiry based multivariable calculus course as well as to derive the characteristics of instructional intervention for enhancing students' argumentation in proof construction activities. Multiple sources of data were collected, students' reasoning in the classroom discussions were analyzed within the Toulmin's argumentation structure and the instructional interventions were gradually revised according to the iterative cyclic process of the design research. The students' argumentation structures presented in the classroom gradually developed into more complicated forms as the study progressed, and the researchers derived the interventions were effective at improving students arguments.

Introduction

One of challenges in undergraduate mathematics classrooms is the shift from traditional teacher-centered and textbook-dominated approaches to new instructional approaches that are student-centered and inquiry-based (Holton, 2001). However, there is a shortage of studies that go beyond basic topics of calculus into areas such as multivariable calculus and differential equations (Rasmussen, 2014). Also, there is a lack of instructional tasks developed for inquiry-based learning (IBL) and a lack of research dealing with classroom interaction and the instructor's role in multivariable calculus teaching/learning. This study attempts to develop an inquiry-based multivariable calculus course and derive the characteristic of instructional interventions for enhancing students's argumentation. The complexity of an argumentation structure depends on the reaction between arguments of the protagonist and critical responses of the antagonists. The complexity of an argumentation structure grows as the discussion is more active (van Eemeren et al. 2007). Thus, the argumentation structure analysis can serve as a quality criterion for mathematical inquiry through proof construction activities in IBL. Considering that learning in IBL is to learn to act and think like a mathematician, students' change of argumentation structure is a proper criterion for the students' learning in IBL. For this purpose, the researchers adopt an empirical approach to study students' arguments in the classroom, and use Toulmin's argumentation structure (1958, 2003) and the classification of argumentation structures suggested by van Eemeren and Grootendorst (1992) as the frameworks of analysis.

Methods

Over a total period of 14 weeks, the students observed two or three online video lectures (20-30 minutes each) and participated in one face-to-face in-line session (75 min) every week. The class was composed of 18 freshmen majoring in mathematics education majors who had taken the course "Calculus I" as a prerequisite, and a total of five small groups of

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three or four students each were set up for learner-centered discussion during the in-class sessions. Depending on the task at hand, laptops or tablet computers were provided for the students to use for discussion or problem-solving purposes.

Results

The students' argumentation structures presented in the in-class sessions gradually developed into more complicated forms as the study progressed, and the researchers conclude that the interventions were effective at improving students' arguments.

Phase 1

The aim of the week's in-class session was to provide students with the opportunity to observe whether the symmetry of partial derivatives holds for two functions and to examine several aspects of the functions, such as graphs, limits, and continuity, in order to inquire about the conditions that would satisfy the property. In the in-class session, however, the students could not reach the final step, in which they were to suggest their own conjectures about the symmetry of partial derivatives. In some steps, students had difficulty constructing their arguments as the researchers had intended, and the instructor had to directly convey certain mathematical knowledge to students that they were expected to be able to derive themselves. Finally, students could not perform well in the last two steps of the task, and the argumentation structure was also different from what the researchers had expected (Figure 1).

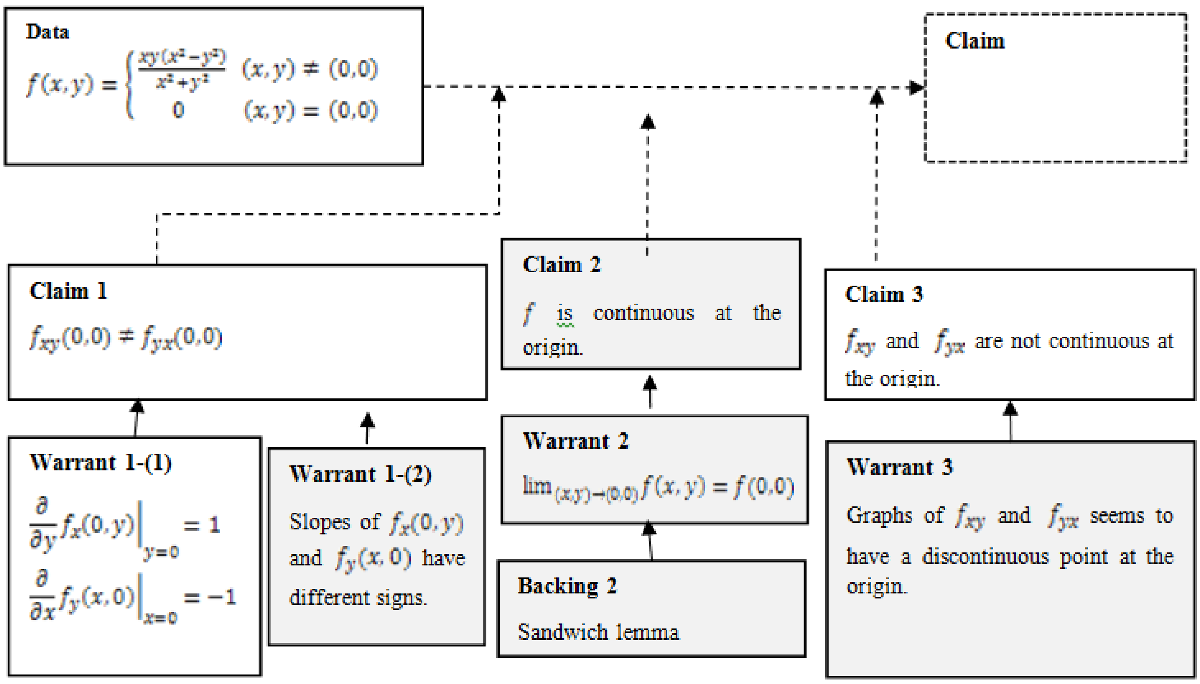

Figure 1: Argumentation structure in Phase 1

In the Figure 1, a solid line is used to represent stages of argumentation that students performed well and a dotted line is used to link parts of the students' argumentation that did not occur in the in-class session; shaded regions indicate parts that the researchers did not anticipate in the design stage or had to change spontaneously during the in-class sessions.

Phase 2

At the end of the in-class session in phase 1, the instructor had explicitly presented Young's theorem and the above lemma and asked students to suggest how it could be proved and to complete the proof of Young's theorem in their reflection journals using the MVT. Student S2 proposed an argument using the MVT twice, and the researchers decided to begin the discussion of how to prove Young's theorem in the fourth in-class session by sharing her idea with her peers. The researchers anticipated that during the session, students would point out some of the problems with S2's proof.

Students proposed three different ways, including S2's proof mentioned above. All proposals were based on the same idea, namely exhibiting the difference in terms of the function and to determine when the concept of limit should be used in the proof. During the whole-group discussion, a multiple argumentation structure focusing on showing the validity of each proof and on comparison between them was observed (Figure 2).

Claim 2

$\exists (c,d) \in (a, a+\Delta x) \times (b, b+\Delta y)$ such that $\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{D(\Delta x,\Delta y)}{\Delta x \Delta y} = \lim_{(\Delta x,\Delta y)\to(0,0)} f_{xy}(c,d) = f_{xy}(a,b)$

Warrant 2

$D(\Delta x, \Delta y) = \Delta y f_y(a+\Delta x, d^*) - \Delta y f_y(a, d^{**})$

for some $d^*, d^{**} \in (b, b+\Delta y)$

Warrant 3

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{D(\Delta x, \Delta y)}{\Delta x \Delta y} = \lim_{\Delta y\to 0} \frac{1}{\Delta y} \lim_{\Delta x \to 0} \frac{f_x(c^*, b+\Delta y) - f_x(c^{**}, b)}{\Delta x}$$

for some $a < c^* < a+\Delta x, b < c^{**} < b+\Delta y$

Warrant 4

Let $g(x) := f(x, b+\Delta y) - f(x,b)$.

Then $D(\Delta x, \Delta y) = g(a+\Delta x) - g(a) = \Delta x g'(c)$

$= \Delta x (f_x(c, b+\Delta y) - f_x(c,b))$

Figure 2: Students' argumentation structure in Phase 2

In this session, the more complicated task of proving Young's theorem was proposed, and a task sequence was implemented beginning with an incomplete solution. It seems that this approach—posing a relatively difficult question incorporating a suggested idea—was more effective than simply providing student with the idea on its own without a specific starting point. By explicitly revealing the controversial point in the proof, the tasks enabled students to suggest multiple warrants for one claim in each small-group discussion, causing the whole-class discussion to result in a multiple argumentation.

Phase 3

In vector calculus, conservative vector fields can be defined in different ways, and most textbooks introduce the definition with several equivalent statements. The task asked students to prove that a potential function exists if the value of line integration is independent of the curve when the starting point and the terminal points are fixed. Researchers design the sequence of the task to construct a new function and examine the function to ensure that it satisfies the definition of potential functions. Although the instructor showed part of the proof to students in the online session to reduce their burden with this unfamiliar and complex task and to improve their concentration, she didn't provide students with individual steps to the proof. In other words, students need to find strategies to develop proofs by themselves.

In this session, the students' proof construction activity was implemented as expected in the HAS, but the instructor had to provide students with scaffolds to help them reach certain sub-claims. Therefore, the students' argumentation structure appeared in the form of the compound argumentation, but showed the slight difference in the shaded regions of the HAS. The shaded regions indicate the instructor's active engagement in the discussion (Figure 3).

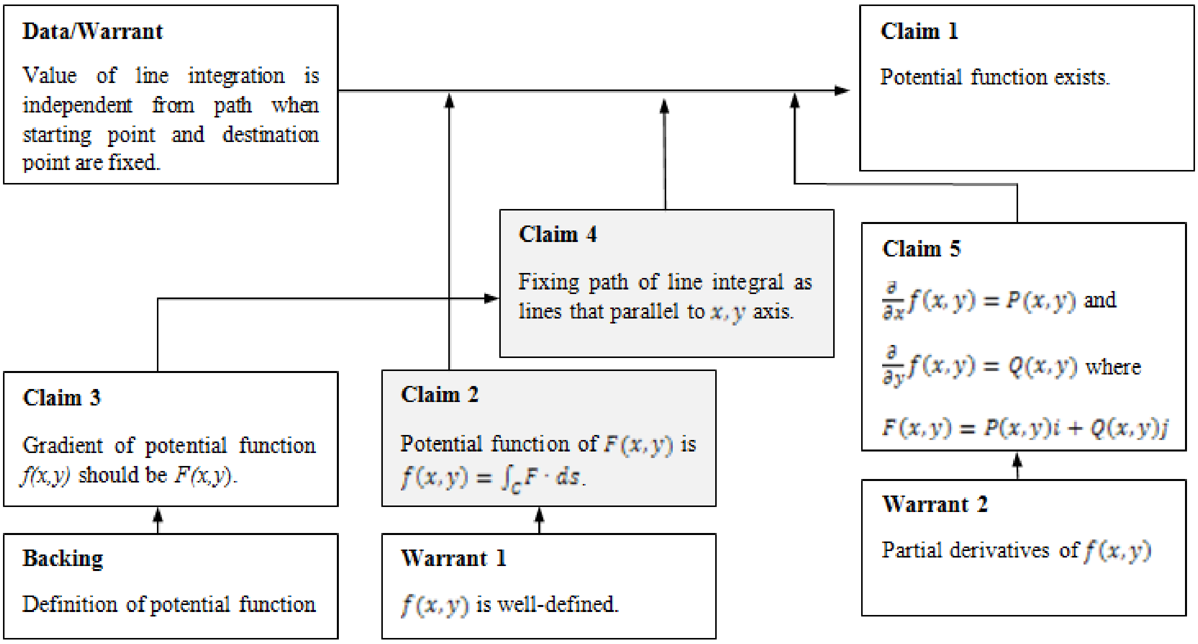

Figure 3: Students' argumentation structure of Phase 3

The main goal of the task in this session was to find and specify new ideas to accurately advance and complete the proof. While the task was described relatively clearly, it is difficult for students as it demanded several complex sub-claims and warrants, and promoted more elaborated arguments. Also, it led to active discussion in small-group discussion and required the instructor's engagements and discussions between small-groups. Therefore, the task contributed to the appropriate environment for IBL so that the students can construct the desired compound argumentation.

This chapter summarizes key findings from the study supported by Center for Teaching and Learning at Seoul National University (Kwon, Bae, & Oh, 2015).

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Pre-service mathematics teachers solve problems in a digital game environment

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Problem-solving in mathematics is considered to be important for developing mathematical thinking. This study focused on problem-solving in a digital game environment. A one-semester course, tailor-made for prospective high school mathematics teachers, was designed and implemented. The course included a chain of digital games in which players were asked to develop a solution strategy. This study examines students' use of the concepts of symmetry, complementary-to-whole and parity as resources while solving problems in a digital game environment. Specifically, the study seeks to evaluate the contribution of this environment (problem-solving and digital games) as a whole to students' mathematical thinking, as exhibited among other things by the differences between their pre- and post- responses to a set of four visual pattern problems. Initial findings indicate that about half of the participants changed the mathematical resources they used to solve the same pattern problems.

Background

"Problem solving provided a way into the joys of doing mathematics and the pleasures of discovery" (Schoenfeld, 2013, pp. 31-32). Problem-solving in mathematics is considered to be important for developing mathematical thinking. Engaging in problem-solving has the potential to develop innovative thinking, encourage creativity and facilitate handling new mathematical challenges (Resnick, 1987). During the past several decades, significant advances have been made in understanding the complex processes involved in problem-solving (Schoenfeld, 2013). One of the challenges associated with problem-solving is persistence—the amount of time students think is appropriate to spend working on mathematics problems. According to Schoenfeld (1992), students who spent most of their time solving short exercises involving mathematical skills expected to solve any problem in a few minutes, leading them to give up on more complex problems they might have been able to solve after only a few minutes of effort. A digital game environment in which players are immersed in a culture and way of thinking has the potential to enhance collaboration and may contribute to students' motivation and persistence (Gee, 2007; Eseryel, Law, Ifenthaler, Ge, & Miller, 2014).

The current research is part of a larger study aimed at describing, analyzing and understanding the problem-solving behavior of pre-service math teachers in the unique environment of digital games. In particular, we focused on possible changes in the problem-solving abilities of prospective math teachers following participation in a one semester course.

Methods

Participants in the study included 41 prospective high school mathematics teachers who participated in an explicitly designed one-semester-long course. The participants were in

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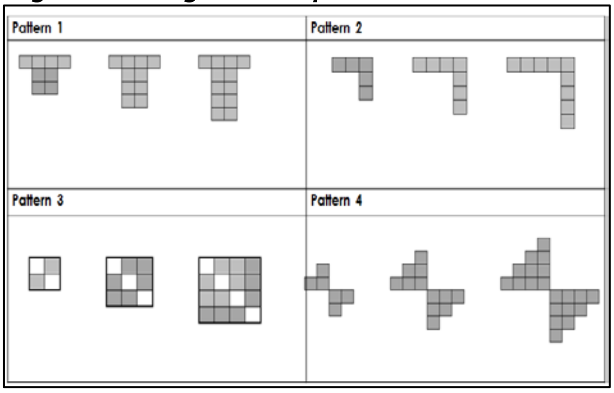
their second or third year (out of four) of studying towards their BA in Math Education and acquiring a teaching certificate. One of the research tools was an identical pre- and post-questionnaire (Figure 1) with four geometric pattern problems that could be represented either by linear $(ax + b)$ or by quadratic $(x^2 \pm x)$ expressions. The question asked with respect to each pattern was: If the pattern continues in the same way, how many gray squares will appear in the n^{th} place of the pattern?

Figure 1: Four geometric patterns



Students' responses were analyzed with respect to Schoenfeld's (1985, 2013) categorization for the analysis of a problem-solving attempt: a) the individual's knowledge/resources; b) the individual's use of problem-solving strategies/heuristics; c) the individual's monitoring and self-regulation/control; and d) the individual's belief systems. In this paper we focus on the first category – resources.

Findings

Students' solution resources were extracted from their responses and were categorized based on Hershkowitz, Arcavi and Bruckheimer (2001): *numerically driven solutions*—solutions that ignore the visual representation—and *visually driven solutions***.** In the article by Hershkowitz et al. (2001) the categories were specific to the presented problem. We extended the categories and classified different generic strategies (Table 1). As shown in Table 1, the participants used a variety of resources that were based on numerical or visual considerations.

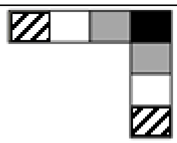
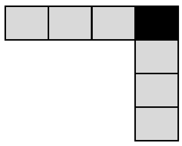
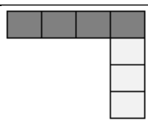
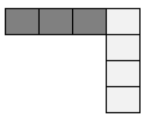
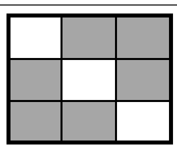
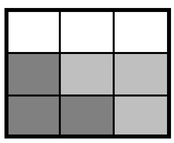
Table 2 shows the distribution of students' use of resources in the pre- and post- questionnaires. As can be seen in the righthand column, the number of students who were not able to generalize the patterns or did not show the solution strategy decreased dramatically. The percentage of students who used numerical considarations as resources decreased, while the percentage of students who used visually driven resources increased. For example, students who solved Pattern 4 using the numerical resource of looking for a numerical pattern between two sequences of numbers were likely to use a variety of visually driven solutions in the post-questionnaire.

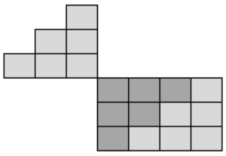
If we compare the solution resource employed by each individual student for each task in her/his pre-questionnaire to the resource student employed in the post-questionnare, we find that for the first and second patterns that were based on changing linear phenomena, 56% and 63% of the students respectively changed their solution resource. For the third and fourth tasks, which were based on changing quadratic phenomena, 34% and 66% of the students respectively changed the resource on which their solution was based. Also, the analysis of student consistency in using the same resource for all four tasks in the post-questionnaire revealed that only 3% of the students used numerical resources.

Summary and discussion

After participating in a semester-long problem-solving course in a digital game environment, students were more likely to use the context of the problem in their problem solving. The geometry pattern problems were given in a visual context, which on the pre-questionnaire was ignored by more than 40% of the students, who used only numerically driven solutions.

Table 1: Categories for students' solution resources

<table>
<tr><td></td><th colspan="2">Category</th><th colspan="2">Example</th></tr>
<tr><td rowspan="3">Numerically driven solutions</td><td colspan="2">Using formula to find the n^{th} term of a sequence.</td><td colspan="2">Example of solution to pattern 1:
"It is clear that this is an arithmetic sequence with d = 2. It was difficult to find a1 in order to fit to the index"
$an = \ 6 + (n-1) \times 2$</td></tr>
<tr><td colspan="2">Looking for numerical pattern between two sequences of numbers.</td><td colspan="2">Example of solution to pattern1:
I was looking at the examples and found the pattern.
(the questionnaire showed a numerical table: age, number of gray squares)</td></tr>
<tr><td colspan="2">Decomposing the sequences of numbers into a variant part and an invariant part.</td><td colspan="2">Example of solution to pattern1:
I was looking for a pattern that grew each time by 2, so by using 2x and according to the first age I found the constant (4)</td></tr>
<tr><td rowspan="4">Visually driven solutions</td><td rowspan="3">Decomposing the visual patterns into parts</td><td>A variant part and an invariant part</td><td></td><td>1 (the square at the corner) + 2 (the number of the additional squares) × age</td></tr>
<tr><td>Symmetry</td><td></td><td>Symmetry between the vertical and horizontal sections and another cube at the edge.</td></tr>
<tr><td>Other</td><td>
or
</td><td>The row is n+1 and the column (excluding the top square) is n.</td></tr>
<tr><td colspan="2">Complementary to a whole</td><td>
or
</td><td>I calculated the area of the square, and subtracted the diagonal.
or
If we connect the gray squares on either side we will get a rectangle whose width = n and height = n minus 1.</td></tr>
</table>

	Combination of visual solutions		***You can see symmetry; the number of gray squares covers a rectangle whose sides are two consecutive numbers.***
	Other		***Visually I saw the pattern.***

Based on the limited findings presented here, we can say that indeed participation in the explicitly designed course contributed to students' mathematical thinking. We consider the numerically driven solutions as ignoring a meaningful aspect of the problem at hand, while we see the visually driven solutions as an indication of students' thinking. Since no such patterns were part of the tasks students engaged in during the course, they had to think about these patterns on their own.

Table 2: Distribution of students' solution resources

%		Numerically driven solutions			**Visually driven solutions**						Failed/ Didn't indicate solution process
					Decomposing the visual patterns into parts						
		Using formula	**Looking for numerical pattern**	**Decomposing the sequences into a variant part and an invariant part.**	**A variant part and an invariant part**	**Symmetry**	**Other**	**Complementary to a whole**	**Combination of visual solutions**	**Other**	
Pattern 1	Pre	17		38	36	2					7
	Post	15	10	12	61	2					
Pattern 2	Pre	16	17	9	7	12	29				10
	Post	7	7	12	10	35	15			12	2
Pattern 3	Pre		37					48			15
	Post		24				2	72			2
Pattern	Pre		41			5		14	20		20
	Post		24			12		39	21	2	2

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Standpoints on elementary mathematics

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Klein's beautiful vision of viewing elementary mathematics from an advanced standpoint inspired generations of mathematicians who have developed an interest in the school mathematics. The recent MET II report recommended that prospective high school teachers take three courses on high school mathematics from an advanced viewpoint. In this talk we attempt to tease apart three components of Klein's perspective, which we call the advanced perspective, the higher perspective, and the technological perspective. We hope to raise questions that serve for a disciplined discussion in mathematics departments on how to implement the MET II recommendation.

Advanced or Higher?

The title of Klein's book in English is "Elementary Mathematics from an Advanced Standpoint," but it could equally have been translated as "Elementary Mathematics from a Higher Standpoint." Is there a difference? "Advanced" connotes being further along, possibly being in a different terrain altogether. Students of advanced mathematics consider mathematical objects unimaginable to the typical school student; groups, fields, categories, schemes, sheaves. "Higher" connotes a broader horizon on the current terrain rather than a different terrain. A higher perspective enables one to see how everything fits together. It is possible to have advanced knowledge of mathematics without having any perspective on high school mathematics at all.

Klein takes both perspectives in his book. In his account of arithmetic, he sees the subject as being unified by the properties of operations, a view which can be obtained from a higher standpoint. He describes ways in which students might approach the commutative law for multiplication (using arrays) or the rules for multiplying negative numbers (using an area model to visualize $(a - b)(c - d)$). He also describes various advanced views of arithmetic, for example as an axiomatic system where the main concern is consistency. He points out that formal systems have a fundamental problem, the "application of these laws to actual conditions." By this he means that there is no guarantee that a formal system concords with the real world of quantities experienced by students. In this way the advanced perspective can sometimes be quite disconnected from the experience of teachers and students. By contrast, the higher perspective should collect and connect those experiences, just as the view from a mountaintop collects and connects many narrower views.

The Role of Direct Experience

A difference between the mathematics of elementary school and the mathematics of secondary school is that in elementary school students can have a direct encounter with the numbers and operations they are studying. When they study whole numbers they can count

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings.* Kassel, Germany: Universitätsbibliothek Kassel.
urn:nbn:de:hebis:34-2016041950121

things and arrangement them in various ways. When they study fractions they can measure lengths or areas. As students advance into secondary school the mathematical objects they study become less and less accessible to direct experience.

As an example, consider the treatment of the logarithm in secondary school mathematics. Students encounter logarithms as they were first encountered in history, as the exponent y in the expression $x = b^y$. The first tables of logarithms, constructed by Napier and Bürgi, used bases b very close to 1, in order to make it possible to work with only integer values of y. As Klein says, Napier and Bürgi "grasp[ed] the thing by the smooth handle." Working with non-integer values of y raises many difficult questions. The definition of b^y when y is rational is subtle, and the extension to irrational numbers is even subtler. One way of viewing this material from an advanced perspective is to define the natural logarithm as a definite integral and then to define the exponential as the inverse function, and verify that the laws of exponents are satisfied. This is a long way from the original intuition of b^y as a power. This is a case where the advanced standpoint occupies different terrain.

The Role of Technology in Providing Direct Experience

The metaphor I have used for distinguishing the higher perspective from the advanced perspective is very much a metaphor from the technological era. We look down on landscapes from airplanes, and we see satellite pictures of the earth. Technology also has a role in providing students with direct experience of advanced concepts. The mathematical subtleties notwithstanding, students can use technology to compute and experiment with exponential and logarithmic functions, to graph them, to explore their properties, to model with them. Technology allows students to work directly with sophisticated mathematical objects, to have those objects become real for them. To a certain extent it allows them to circumvent advanced mathematics, which can be both a good thing and a bad thing. In the case of logarithms, it would be a good thing if technology helped students see a broad picture of how all the pieces fit together, for example the relationship between properties of exponents and properties of logarithms, or the complementarity between the rapid growth of exponential functions and the slow growth of logarithmic functions. Such a collection of connections might constitute what we have called a higher perspective on the topic of logarithms. On the other hand, the naturalness of the base e for the natural logarithm is a mystery without the advanced perspective.

Klein himself was fascinated by technology. His book is full of computing machines and computing devices, and he was well aware of their potential to promote mathematical explorations. He was interested in how their mechanisms reflected the underlying mathematics. By contrast, technology in education today is often viewed as a black box, a tool for exploration. The analog of Klein's preoccupation would be an interest on the part of mathematicians and educators in the use of computer programming in the teaching of mathematics. A recent project of Al Cuoco et al at EDC aims to study the relationship between a computer programming course and mathematical habits of mind.

Which Standpoint for Prospective High School Teachers?

The purpose of this discussion is to provide some framework for discussing the question of what mathematics prospective high school teachers should know. In the United States, future high school teachers often take courses designed for the mathematics major. Since these courses are also designed to prepare students for graduate work in mathematics, they are often courses that look forward to more advanced topics rather than courses that reflect backward on high school mathematics. The 2012 MET II report recommends that high school teachers in the United States “should be required to complete the equivalent of an undergraduate major in mathematics that includes three courses with a primary focus on high school mathematics from an advanced viewpoint.” This report is aimed at departments of mathematics, who generally have responsibility for the content knowledge of high school teachers. It would be helpful in implementing this recommendation for mathematicians in those departments to conduct a disciplined inquiry into the three standpoints on their subject matter discussed here: advanced, higher, and technological.

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Exploring and overwriting mathematical stereotypes in the media, arts and popular culture: The visibility spectrum

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I discuss an analytical and pedagogical tool, the "visibility spectrum", which can be used to determine the degree and quality of presence of mathematics and mathematicians in the media, arts and popular culture. The spectrum comprises six levels: "invisibility"; three types of "exotic presence" ("auxiliary", "vilification", "admiration"); "political correctness"; and, "normalization" and has been adapted from media and cultural studies which trace how/whether social groups that are often under/inappropriately represented gradually gain a more acceptable kind of visibility. Here I demonstrate the levels of the spectrum and offer evaluative evidence of its use in teaching students majoring in Education and largely destined to become primary teachers.

Introduction

The relationship between mathematics and students is often tantalised by perceptions of tedium, difficulty, lack of creativity, elitism and unsociability. *Outside* school one influence on young people's attitudes (and choice of field of study) can be in the ways mathematics and mathematicians are portrayed in popular culture. While our first priority needs to be with improving students' experience of mathematics *within* school, we also need to develop systematic ways of working against stereotyping and towards engineering more favourable, and accurate, images. *Within school* we need to openly address these images: question the inaccurate, undesirable ones, and make the most of the rest. *Outside school* we need to work more closely and systematically with the often well-intended, but not always best-equipped, 'outsiders' who create those popular images.

The preparation of teachers rarely equips them for this complex task. Here I draw on materials collected and analysed for research purposes as well as my teaching a module entitled *Children, teachers and mathematics: Changing public perceptions of mathematics* to undergraduates majoring in Education, and largely destined to become primary teachers. The research, and the associated module, explores questions such as "what are the dominant public perceptions of mathematics and mathematicians?" (thereafter Q1) and "if we were to work towards overwriting stereotypical images of mathematics and mathematicians, what images would we replace them with?" (thereafter Q2). The study and the module activities make extensive use of an analytical and pedagogical tool, which I call the *visibility spectrum*, and which can be used to determine the degree and quality of presence of mathematics and mathematicians in the media, arts and popular culture.

One observation that underlies the work that I draw on here is that the timing for considering Q1 and Q2 is particularly ripe, given the increasing presence, in recent years, of mathematics and mathematicians. To address Q1 and Q2, I deploy the *visibility spectrum* to exam-

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ine media, arts and popular culture excerpts to trace dominant discourses on mathematics and mathematicians (from less to more desirable ones). To address Q2, I also draw on accounts of mathematical experience, collected from learners across educational levels and ages to propose a more desirable, and accurate, image of mathematics.

In what follows I outline the *visibility spectrum* and young people's often ambivalent relationship with mathematics. I then sample the dominant ways in which mathematics education, mathematics and mathematicians are portrayed in the media, arts and popular culture. I conclude with an indication of an alternative, and more accurate, portrayal towards which we, as mathematics educators and mathematicians, can work more systematically.

Tracing discourses on mathematic(ian)s: the *visibility spectrum*

The *visibility spectrum* is a theoretical construct – which I conceived originally as an analytical tool, and more recently I have been deploying also as a pedagogical tool – that can be used to determine the degree and quality of presence of mathematics and mathematicians in the media, arts and popular culture. The spectrum comprises six levels: "invisibility" (0); three types of "exotic presence" (1-3: "auxiliary", "vilification", "admiration"); "political correctness" (4); and "normalization/acceptance" (5). For example, in the context of film, the *visibility spectrum* can be used to trace: absence of portrayals of mathematics and mathematicians in certain film genres (0); evidence of *some* visibility of mathematics and mathematicians as a form of otherness that is either auxiliary and largely insignificant (1), or associated with villainry (2), or lined with awe at the extraordinary mathematical ability of certain characters, fictional or historic (3); evidence of a politically and educationally correct, often deliberately positive portrayal, particularly of important mathematical figures (4); evidence of a normalised and natural portrayal (5). A few examples follow.

The visibility spectrum emerged out of works in media and cultural studies (e.g. Fiske, 2010) that examine portrayals of other 'differences' (e.g. in relation to class, gender, race, ethnicity and sexual orientation) and is predicated on two (not fool-proof yet) assumptions: that deploying a construct that emerged out of studies of the various 'differences' listed above is valid in the context of mathematics education; and, that there is a direct influence of popular portrayals of mathematics and mathematicians on the ways in which young people relate to mathematics. Works such as Picker & Berry's (2000, the data of which is sampled in Fig. 1) have started to explore this assumption.

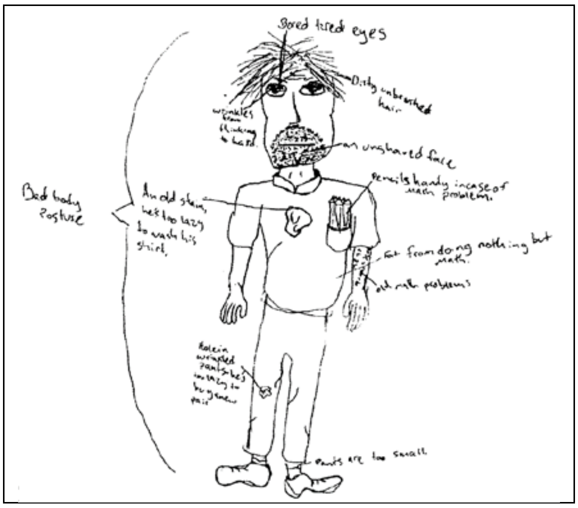


Fig. 1. Pupil portrayal of mathematician from Picker & Berry (2000)

My use of the *visibility spectrum* is two-fold: to examine media, arts and popular culture excerpts to trace dominant discourses on mathematics and mathematicians (ultimate aim: identify, and engineer, trajectory from less to more desirable ones); and, towards analysis of learners' accounts of mathematical experience (ultimate aim: to engineer trajectory towards more favourable experiences and perceptions). This use is aimed across educational levels and ages and has the potential to become a point of synergy between mathematics educators and mathematicians (Nardi, 2014). Theoretically these aims resonate with broader aims often found in the area of cultural studies in (mathematics) education. As the Editors of Jubas, Taber & Brown, (2015) note in the preface of their *Transgressions: Cultural Studies and Education* book series:

> *'… cultural studies scholars often argue that the realm of popular culture is the most powerful educational force in contemporary culture. […] Educators […] must understand these emerging realities…[…] Without an understanding of cultural pedagogy's (education that takes place outside of formal schooling) / role in the shaping of individual identity – youth identity in particular – the role educators play in the lives of their students will continue to fade.' (2015, Preface)*

In this their words resonate with works on affect and meta-affect in mathematics education (e.g. DeBellis & Goldin, 2006):

> *'Meanings provided young people by mainstream institutions often do little to help them deal with their affective complexity, their difficulty negotiating the rift between meaning and affect. School knowledge and educational expectations seem as anachronistic as a ditto machine […]. […]school knowledge and educational expectations often have little to offer students about making sense of the way they feel, the way their affective lives are shaped.' [ibid.]*

They conclude:

> *'In no way do we argue that analysis of the production of youth in an electronic mediated world demands some "touchy-feely" educational superficiality. What is needed in this context is a rigorous analysis of the interrelationship between pedagogy, popular culture, meaning making, and youth subjectivity.'*

This study takes this cue and brings it into mathematics education. For example, in Chapter 4 in (Jubas *et al.*, 2015) entitled *Teachers on Film: Changing Representations of Teaching in Popular Cinema from Mr. Chips to Jamie Fitzpatrick,* Tony Brown traces a shift in film narratives about teaching and teachers (shepherd, guardian, hero, social mediator and more recently as agent of school/learner failure). The study I discuss here asks: What are these narratives for mathematic(ian)s? Are they shifting? And, if so, how? Within mathematics education questions such as these are akin to the work by Moreau, Mendick and Epstein (e.g. 2010) and their ESRC-funded study of identity shaping forces of popular culture (with a focus predominantly on gender and to some extent class). Here is their definition of *identity* and *identity work* that I draw on in the present study:

> *'…imagining the self as a complex and contradictory space in which discourses […] work and are worked […] identity as something always in process and never attained and so as requiring constant effort. To capture this I use the phrase 'identity work' […]. Another way of thinking about this is to read identity as a verb rather than a noun, something that we do, and are done by, rather than something that we are.' (Mendick, 2005, p.205)*

Within mathematics education there is a small body of work that describes portrayals of mathematic(ian)s and relationship between these and images held by young people (e.g.

Gadanidis & Scucuglia, 2010; Lim, 1999). For the purposes of this study I endorse Lim's (1999) definition of 'image of mathematics' as follows:

> *'a mental representation or view of mathematics, presumably constructed as a result of social experiences, mediated through interactions at school, or the influence of parents, teachers, peers or mass media. Also includes: 'all visual and verbal representations, metaphorical images and associations, beliefs, attitudes and feelings related to mathematics and mathematics learning experiences. (p.2)*

Young people's ambivalent relationship with mathematics

A pragmatic and theoretical origin of the work I discuss here lies in the fact that, at least in the UK where most of this work is being conducted, young people's relationship with mathematics is overall negative – particularly as they enter adolescence. For example, Nardi and Steward (2003) described secondary pupils' extensive and often quiet disaffection from mathematics as T.I.R.E.D. (characterized by *Tedium, Isolation, Rote Learning Practices, Elitism and Depersonalisation*) and Brown, Bibby and Bibby (2008) traced how this relationship seems to drive students away from undertaking mathematics studies at post-compulsory level. Again at least in the UK, the often morbid coverage in the media of issues related to the ways in which young people experience – and perform in – mathematics in school exacerbates and, some may say, perpetuates largely negative images. Two key characteristics of this portrayal (see, for, example, Fig.2) are: the deficit discourses on mathematics *teachers* and *teaching*; and, the stereotypical, and largely unfounded portrayals of *mathematical ability*.

The Telegraph

10 November 2011

Children failed by 'flawed' maths lessons, warns examiner

Mark Dawe, chief executive of OCR, one of Britain's biggest exam boards, said growing numbers of young people struggled to function in further education or the world of work after failing to "acquire the maths skills that society demands" at school.

He suggested that the existing curriculum was unable to cater for children with different needs, including the very brightest at one end and those that struggle with the basics at the other.

Currently, almost half of 16-year-olds fail to achieve grade C at GCSE, with just 15 per cent studying maths beyond that level.

It is also feared that as many as a quarter of economically active adults are "functionally innumerate".

Speaking ahead of a conference on maths education today, Mr Dawe said lessons "will always be flawed until schools, universities and employers agree on what maths skills they really want from young people".

"Maths means different things to different people," he said. "Some say it's all about numeracy – the facility to add, subtract, multiply and divide whole numbers, with perhaps, a little bit of percentages thrown in – whereas others equate maths with arithmetic – the art of calculation.

"Some believe 'real maths' helps unpick the secrets of the universe. Whichever it is, the system clearly isn't delivering."

Speaking before a conference at the Royal Institution in London, he said: "Too many students do not acquire the maths skills that society demands which means they can't enjoy mathematics or take it into further education, the workplace or use it in everyday life."

OCR has now set up a new "maths council" to gather views on the future direction of the subject.

Fig. 2. Media excerpt portraying typical perceptions of mathematics teaching.

Portrayals of mathematics and mathematicians: a sample

The bulk of the media, arts and popular culture excerpts that have been the focus of the work discussed here (so far the analysis in focused on: film; anglo-centric, mainstream examples; initial phase of the study time span: 12 years (1997-2009); current time span: 2 years (2013-2015)) have been found to treat mathematical ability as madness, or at least as strangeness – levels (2) or (3), in the language of the *visibility spectrum.* Well-known examples, include treatment of this madness, or strangeness, rather variably: as clinical (*A Beautiful Mind*, Fig. 3), of a more poetic / metaphysical nature (*Pi*), part of a web of complex familial, social and professional relationships (*Proof*, *A Serious Man*) and often embodied by a young and attractive male (*Good Will Hunting*). The strangeness/madness of the mathematically-related characters in these films is palpable. Crucially – and without a hint of questioning – their mathematical ability seems to go hand in hand with this strangeness/madness. Out of their stories mathematics emerges as a preoccupation of the few (and the rather odd): if there is a mathematician in sight then they must be of the genius type. And (most often) *he* must be mad. Often too, the narrative in these films is underlain by two, in my view, problematic juxtapositions: between *intelligence* (as devious artifice, and with a propensity for evil) and *ignorance/stupidity* (as doe-eyed innocence and natural, unspoilt goodness – see, e.g., *Forrest Gump* and TV sitcom character Joe in *Friends*); and, between *action* and *theory* (mathematical 'geniuses' dragged out of their ivory towers by no-nonsense, street-wise action men, and occasionally women, so that their 'genius' can finally become of use, see, e.g., *Jurassic Park, Numb3rs*). It is in this cultural ambience that mathematics educators often find themselves defending their subject to prejudiced audiences and most people – often including not only pupils but parents and, even, teachers – casually admit to *mathophobia*. It is in this cultural ambience that dislike of and disengagement from mathematics emerges as natural and socially acceptable.

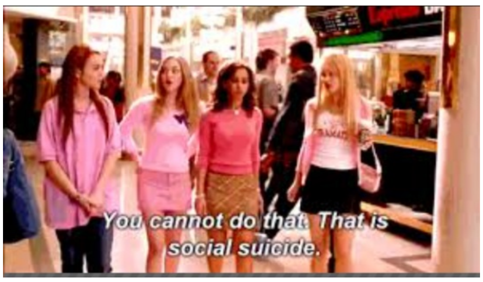
Example 1 (Fig. 3): A Beautiful Mind (Goldsman, 2001). An account of the life of mathematician John Nash, Nobel prize winner and creator of Game Theory. John Nash suffered severe mental health problems throughout his life. Here is a quotation from the trailer voiceover:

> *The extraordinary gift, that set him apart, would push his mind beyond its limits.*

Fig. 3. Portrayal of mathematical ability as otherness/madness, A Beautiful Mind.

Example 2 (Fig. 4): Mean Girls (Fey, 2004)

> ***Cady****: I think I'm joining the Mathletes.*
>
> ***Regina, Gretchen, Karen****: No! No, no!*
>
> ***Regina****: You cannot do that. That is social suicide. *Damn*! You are so lucky you have us to guide you.*



***Damian**: [reading Cady's class schedule] Health, Spanish... you're taking 12th Grade Calculus?*

***Cady**: Yeah, I like math.*

***Damian**: Eww. Why?*

***Cady**: Because it's the same in every country.*

***Damian**: That's beautiful. [to Janis] This girl is deep.*

Fig. 4. Portrayal of mathematical ability as 'social suicide', Mean Girls.

Towards a more desirable, and accurate, image of mathematics

I would now like to put forward a more appealing, and in my view accurate, portrayal of mathematics as offered by learners, doers and users of mathematics from across educational levels, and based on evidence and testimonials from studies that I have been involved with over the years: research mathematicians, (Nardi, 2008); mathematics undergraduates (Nardi, 1996); aforementioned secondary students (Nardi and Steward, 2003), primary pupils (Ainley, Pratt and Nardi, 2000). Mathematics emerges out of these testimonials as a powerful way of reasoning, expressed in technical-yet-effective language, and as a rewarding intellectual pursuit and preoccupation. A bold portrayal of mathematics involves, as per participant mathematician in (Nardi, 2008) choosing to 'stop shying away from the nature of our subject'. It also involves moving away from what Nardi and Steward (2003) called the 'mystification-through-reduction' (p.362) of school mathematics, the attempt to 'simplify' mathematical thinking by converting into execution of cues and procedures – and thus embedding it even more irrevocably into students' experience as a 'hierarchical game' (p.362), played only by those who can, and leaving out the longing for understanding, hence intellectual satisfaction and ultimate enjoyment of the subject.

My overall claim is that mathematics does not need to try to be, as a school subject as well as in popular perception, what it is not; nor to suppress what it actually is. In fact all mathematics needs to do is celebrate, publicly and dynamically, its true nature: alienation from the 'nature of our subject', 'dumbing it down', forcing it into an artificial and ultimately unconvincing straightjacket of 'accessibility' and 'relevance' drains the life out of it and detracts from its most crucial, and attractive, characteristics. There are three clusters of activity that I would like to report in order to sample developments in this direction:

(1) *Theory:* This concerns the emergence of science communication as a discipline in its own right and particularly Emergence of analyses of public (colloquial) and private (literate) discourses of mathematics. One example is Barwell's (2007) analysis of mathematicians' talk (in this case during a live radio broadcast about the Poincaré conjecture). The works that I exemplified in the earlier parts of the paper and offer analyses of popular culture as (mathematics) pedagogy and extending and refining works on meta-affect are further examples.

(2) *Public Engagement*: This concerns activities that aim to draw non-mathematical audiences into the world of mathematics and into considering the possibility of mathematical studies. One example of these are – in my institution – the MAUD events (Maths at Uni Days,

since 2006) a collaboration between the Schools of Education and Mathematics, also with the Further Mathematics Centre hosted at UEA. Aim: to showcase the importance and appeal of mathematics as well as its capacity to open windows to a wide range of professions (Bills, Cooker, Huggins, Iannone, & Nardi, 2006).

(3) *Teaching*: This concerns designing and offering modules that use research-informed analyses to build towards a more desirable, and accurate, portrayal of mathematic(ian)s. Aim: to change perceptions of mathematics and mathematicians held by key stakeholders in the teaching and learning of mathematics, e.g. prospective teachers. This also concerns designing classroom interventions that alert students to (and challenge), portrayals of mathematic(ian)s in the media, arts and popular culture. As an example I would like to put forward an undergraduate module that I have been teaching in my institution since 2012 (*Children, teachers and mathematics:* Changing public perceptions of mathematics, BA Education, Year 3, Autumn Semester). I quote from the flyer distributed to students who are considering taking the module as an option (student uptake has been rising from 25% to 75% over three runs of the module):

> *'This module explores a range of issues that relate to young children's learning of one of the most important, yet notoriously feared and misunderstood, subject: mathematics! […] We aim to share some of the excitement experienced by those who love mathematics – enthusiastic teachers, university mathematicians and other professionals – but we also examine some of the key challenges that young children face when they engage with mathematical learning in primary school. We investigate where the social and psychological 'stigma' of mathematics comes from – the fear that prevents many people from building a good relationship with mathematics. We juxtapose this 'stigma' with results of neuroscience that show that mathematical thinking is quite natural; in fact that mathematical ability is innate to all human beings! We also juxtapose these research findings with examples from popular culture (TV, films, pop music) and the arts that seem to perpetuate largely 'math-o-phobic' images. We consider how education, particularly in the crucial years of primary school, can work against the tide of such images and introduce children to the creativity and excitement of mathematics!'*

Following 10x2h lectures, the module offers 10x2h seminar pre-set activities which include: Media excerpt tasters; Student accounts of relationship with mathematics; Student showcase of media excerpts; Reflections on the mathematics curriculum I (early and upper primary years); 2-minute Maths Pitches (Student choices); Analysis of classroom incidents (mathematical, social and affective perspectives); Student analysis of own selection of media, art and popular culture excerpts; Student analysis of portrayals of mathematic(ian)s on film (five pre-set films); Student *Debunking Myths About Maths (*Kogelman & Warren, 1978: Innate, Male, Introvert, Burn Out, Uncreative); Student 3-5 mini maths lessons.

In conclusion: Our first priority needs to be with improving the students' experience of mathematics *within* school. We also need to develop systematic ways of working against stereotyping and towards engineering more favourable, and accurate, images. So, *within school* we need to openly address these images and question the inaccurate / undesirable ones. Most of all, we need to make the most of accurate and desirable images. *Outside school* we need to make our within school efforts better known, understood and appreciated by the general public. Because one influence on young people's attitudes (and choice of field of study) originates in images of mathematics and mathematicians in the media, arts and popular culture, *outside* school we also need to work more closely and systematically with the often well-intended, but not always best-equipped, 'outsiders' who create those popular images.

The moment seems to be apt for this image-shifting enterprise. I quote from www.BoxOffice.com, the marketing strategy put together by the producers and distributors of Ridley Scott's *The Martian* (2015) which at the time of writing had grossed about half a billion US dollars in the box office and was also collecting several accolades for its script, direction, acting and visual effects – see trailer at https://www.youtube.com/watch?v=ej3ioOneTy8, with astronaut Mark Watney, played by A list Hollywood actor Matt Damon, is 'doing the math' (many times). In the film's marketing strategy, engagement with science is a 'draw, not a deterrent'. The film presents 'relatable, cool science', 'surprising plausibility' and 'the technical details keep the story relentlessly precise and the suspense ramped up'. Commentators praise 'Matt Damon's everyman persona infused by terrific wit', its 'humor mixed with smarts' and how it 'made science cool again'. So, science has its *visibility spectrum* level 5 item… *Time, perhaps, for mathematics to acquire its own too?*

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How do pre-service teachers experience math didactics courses at university?

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In Germany teacher students often express dissatisfaction with the current institutional arrangements of teacher training. To gain a better understanding of this phenomenon, first, the teacher training system in Germany will be briefly described. In doing so, the position of math didactics and its discipline culture will be taken into account in relation to mathematics and educational sciences. Second, the experiences of teacher students in university courses will be discussed by taking into consideration their own concept of teacher professionalization. The findings are exemplified by the case of the teacher student Anna.

Teacher training in Germany

Teacher training in Germany is split in two phases; the first phase is situated at university and the second phase takes place in school and in specific seminars outside the university. During a bachelor's and following master's degree students shall first build up a substantial base of theoretical knowledge about mathematics and teaching, before going to practical training in the second phase. The federal states of Germany organize their teacher education differently, but mostly follow a structure similar to the one of Lower Saxony presented in the following.

When enrolling at a university, teacher students choose two school subjects they want to teach. Depending on the school type for which they want to become teachers, teacher students choose to study a specific bachelor's and master's degree program: primary and lower secondary education or vocational school education. Future secondary school teachers begin their studies in an interdisciplinary bachelor degree program and continue their studies doing a master degree in education. These different study programs vary in their structure and focus. In primary and lower secondary teacher education programs more weight is put on pedagogical subjects, while in secondary school teacher education programs the mathematical requirements are higher. Our contribution focuses on the first phase of primary, lower secondary and vocational school teacher education.

These mathematics teacher degree programs include the study of elementary math, math didactics, educational sciences, the other chosen subject and its related subject matter didactics. Each of these different fields of study is shaped by its specific discipline culture.

Discipline cultures

The different academic disciplines can be distinguished by their epistemological structure and focus. According to these two axes, Becher (1989) developed a taxonomy of four different types of discipline cultures: Hard pure, hard applied, soft pure and soft applied.

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- Hard and pure disciplines aim at general propositions, try to find recurrent patterns and seek simplicity in theories. It is possible to split the knowledge base into traditional subthemes. Mathematics can clearly be identified as hard and pure discipline.
- Hard and applied disciplines draw on hard and pure disciplines for their knowledge base, but have a strong focus on professional practice and therefore application (e.g. Engineering).
- Soft and pure disciplines aim to understand phenomena in their complexity and a diversity of theories makes it difficult to clearly identify subthemes (e.g. sociology).
- Soft and applied disciplines draw on diverse theories, with a strong focus on their implications for professional practice. Educational sciences fall into this category, relying on theories from several different disciplines and being influenced by different discipline cultures.

Math didactics can be described as an interdisciplinary discipline ranging in between its contributory disciplines mathematics and educational sciences. These two disciplines are located at opposite corners of Becher’s two-dimensional taxonomy system. Math didactics therefore can be perceived from different angles and foci. On the one hand, math didactics is a scientific discipline trying to merge several theoretical influences; on the other hand, math didactics is an applied subject and is complicated by vocational pursuits. Especially in the context of teacher training, math didactics is presented as a subject providing a bridge between mathematics and educational sciences. It is the one subject that is specific to mathematics teacher training in university context.

The different types of discipline culture affect the curriculum structure, teaching and learning in the specific disciplines. Math didactics is therefore framed by its reference disciplines. Neumann (2003) identifies the epistemological characteristics of teaching and learning in these reference disciplines as follows:

- Hard and pure disciplines follow a curricular structure, whose organization is linear, hierarchical and cumulative. The focus lies on understanding theory in depth, reasoning and discipline-specific skills. It is hallmarked by assessments in form of examinations.
- Soft and applied disciplines are structured dominantly in seminars addressing research from different paradigms. Because of its focus on the later application the aim is broad knowledge relevant for practice related skills. These are assessed in form of essays.

These show that teacher students have to handle different epistemological views and requirements in math didactics courses.

How do pre-service teachers experience math didactics courses at university?

The preliminary results presented in the following are part of an ongoing qualitative study addressing learning experiences of pre-service mathematics teachers in the institutional context of university teacher education programs in Germany.

It’s not very surprising that teacher students’ rationales are diverging with regards to the question of which focus math didactics courses should have and which knowledge math didactics is supposed to provide. In the following we explore this rationale further and examine how it is linked to discipline cultures. For this purpose we don´t see discipline cultures as a behavior-determinant, but explore according to the subject-scientific approach (Holzkamp, 1985) the meaning-reasoning relations from the teacher students' standpoint. Holzkamp (1985) argues that humans recognise the world from their own perspective and with purpose; reality is interpreted by the subject in connection with her or his experiences and intentions and in view of their perceived "life interests".

Teacher students therefore ascribe meanings to their experience with different discipline cultures in relation to their own concepts of professionalization; this affects their perception of math didactic courses.

Method

Eight pre-service teachers – aiming to be primary school, lower secondary school or vocational school teachers – have been interviewed at the Leuphana University Lüneburg. The pre-service teachers had different educational backgrounds and were at different stages in their studies.

The semi-structured interviews addressed questions about students` concrete learning activities, the students' motivation to become mathematics teachers, perceived support and experienced conditions of studying. After the interview, each student was asked to complete a Mind Map on what she or he considered important for her or his learning. The subjective standpoints of the students were the starting point of the analysis. The data is currently analyzed using a combination of grounded methods (Strauss & Corbin, 1990) and techniques provided by objective hermeneutics (Wernet, 2009).

Preliminary results

In the following, findings are exemplified by the case of Anna. Anna studies in a primary and lower secondary teacher education program and aims to be a lower secondary school teacher. At the time of the interview, Anna had just completed her bachelor's and was about to begin her master's program. She chose to write her bachelor thesis in math didactics.

Significant interaction takes place between teacher students' **views on mathematics** and their experiences in university courses. Anna's experiences with mathematics in school context led her to develop a quite stable view, that doing mathematics will lead to definite solutions. She recounts that she liked mathematics in school because *it was always correct or false, and therefore the grading doesn`t rely on the mathematics teacher's interpretation*. In university courses she relies on sample solutions for learning mathematics, providing her with a secure guideline of what she is supposed to learn. She strongly emphasizes structure in her learning and recognizes her inability to have an overview of the knowledge she is supposed to learn, especially in the beginnings of her study.

The **view on mathematics teaching** is linked to the view on mathematics, but brings in the substantial element of the perception of the later profession. Anna's view on mathematics teaching can be described in terms of the teacher being a guide through mathematics (as she is looking for a guide through mathematics at university), who is in charge of leading all pupils through difficulties in mathematics. Therefore she anticipates providing step by step instructions to her future pupils. For her, the fundamental task of a teacher is taking care of and controlling the learning progress of her future pupils, that leads her to search for the **best-practice** answer.

This eagerness to figure out the **best-practice** for applying the knowledge she learned at university and the **mode of assessment** allows us to understand her **perception of math didactics courses**.

- In her opinion, teacher training should be structured more like an apprenticeship model, more so than the current German model. She would prefer to actually be co-teaching twice or three times a week and imagines benefits for teacher students and the teacher at school. She acknowledges the importance of theoretical knowledge in elementary math courses seeks for (practical) methodical knowledge in math didactics courses.
- Assessment provides her with the security of having mastered learning tasks (guiding function) and therefore plays a crucial role in her personal learning. Her preferred mode of examination is the written exam, in all her different fields of study. She is overstrained by essays in math didactic courses and feels left alone with this task. She cannot picture the purpose of essays in math didactics courses and hence has difficulties figuring out her learning tasks, while writing essays. The rationale of linking different theories and discussing their implications, instead of finding the one solution – which is typical for soft and applied disciplines – remains alien to her.

Hence she primarily expects math didactics to provide her with methodical knowledge. Thus math didactics courses can only partially satisfy her expectations. Anna does not recognize a bridging of and educational sciences neither on a theoretical nor on an analytical level. She cannot integrate (theoretical) methodological considerations into her perception of math didactics and thus cannot integrate practices coming from an educational sciences culture into her own learning in math didactic courses.

Teacher students` experiences in math didactics courses are linked to their own professionalization concepts and can essentially be characterized by their view on mathematics and mathematics teaching. Especially the rationale on assessment in math didactics courses displays a strong discipline culture influence.

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Teaching undergraduate mathematics – reflections on Imre Leader's observations

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Imre Leader is a Professor of Pure Mathematics at the University of Cambridge. In 2013, he was in Singapore as the 10th Singapore Mathematical Society Distinguished Visitor. In an interview published in the Mathematical Medley (Yap, 2013), Leader answered a wide range of questions regarding mathematics and mathematics education at the university level. In this paper, we reflect on some of his observations and distill their implications with a view to framing a research and developmental agenda for teaching undergraduate mathematics for pre-service teachers.

Introduction

Imre Leader is a Professor of Pure Mathematics at the University of Cambridge. In 2013, he was in Singapore as the 10th Singapore Mathematical Society Distinguished Visitor. In an interview published in the *Mathematical Medley* (Yap, 2013), Leader answered a wide range of questions regarding mathematics and mathematics education at the university level. Leader is an excellent mathematician (by this we mean that he does original work in mathematics and publishes actively as a result) and a very engaging lecturer – the first author attended his talk on the Ackermann function to a group of secondary school students and learnt something new as well as saw that some students learnt as well. In addition, he has taught mathematics at Cambridge since 1989 and so his views on teaching university mathematics should give us pause for careful consideration.

In this paper, we reflect on some of his observations and distill their implications with a view to framing a research and developmental agenda for teaching undergraduate mathematics for pre-service teachers.

What kind of mathematics to teach?

Leader talks of two further broad kinds of needs for students who do not intend to become professional mathematicians (Yap, 2013). First, for those who would go on to be using mathematics a lot, such as people going to engineering, physics, chemistry, economics, he recommends that they learn "just ... what they need" (p.3), i.e. the mathematical concepts and formulas necessary for their professions. On the other hand, if they would not be going to use mathematics a lot, such as those going to banking, it seems to him most important that they "learn to think" (p.3), which we interpret to be problem solving skills, logical thinking and some rigour. With the limited time in the curriculum, we do not understand him to mean that the first group would be taught "how to think" as well.

With regard to future school teachers, Usiskin (2001) concurs with the need for relevant mathematics, i.e., the thorough understanding of the mathematics that they will teach in the

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school. Thus, they would seem to fall into the category of non-professional mathematicians who would be using mathematics a lot. However, teachers need to also transmit the discipline of mathematics to their students and so it seems to us that a level between mere users of mathematics and professional mathematicians is justified. Teachers of mathematics at every level should see the big picture in mathematics. They need to know how mathematicians solve unfamiliar problems and so model such behaviour to their students, understand how mathematicians tackle infinity and thus guide their students to not only rely on their real-world intuitions, understand the centrality of proof and so help their students see a distinguishing aspect of mathematics as a subject, and learn how to read and write mathematics and here see that, like other subjects, self-directed learning in mathematics depends very much on personal reading and writing. We name such a product of the ideal pre-service teacher mathematics curriculum as the *mathematician educator*.

What pedagogical approach to take?

Leader describes the Cambridge approach to teaching mathematics is as mainly the traditional lecture-tutorial system: "In the lecture, you present the material. But then, the point of the homework sheets, or example sheets, should be to make the students partly understand the material, but much more to think about the material. Both are really important." (Yap, 2013, p.5) Then he makes the startling assertion: "Maths is hard. Almost every student in almost every lecture is lost after 15 minutes. Maybe they have a vague idea of what is going on, but the details are left out, even for the best students. It's human nature … most of the time they are not following it." (p.5) Remember that we are talking about Cambridge University, an upper bound for ability, so Leader's statements imply that although necessary for first contact with the content, lectures are wholly inadequate for learning and must be supplemented with tutorial questions that require work and deep thinking.

To the question of how often a maths course involves students reading and learning from a textbook, the reply was: "Never ever. So in Cambridge, the maths course is based on the lecture. So the feeling is, if a student has copied out from the board, he will understand it. With a textbook, a student can read it line by line, but not understand in any depth." Leader is very sceptical about students reading on their own before a lecture: "If the student has lecture notes, the student will not concentrate during the lecture. They will daydream, they will say, "Oh I could read it some time." Second thing: the lecture gets much worse. The lecturer thinks "I can be disorganized. I can skip some bits which are in the notes." … thirdly, … we find that when a student has written down stuff in his own handwriting, even if he doesn't understand it, he has some ownership that he wants to learn it. If a student photocopies other students' notes or gets printed notes, he doesn't own it, will never learn it. … I think printed notes are a great great evil. All students want them, and think, "Of course they help. How can they not help you?" But they are very bad, for these reasons." (Yap, 2013, p.5) Such comments would seem to undermine efforts to teach students to read (see for example, Alcock, Hodds, Roy & Inglis, 2015; Tay, 2001, 2014) and Leader adds: "That's great if the students have already, in advance, read the day's lecture notes. But of course real life isn't like that. Students never read beforehand. It's always a disaster. It's just human nature." (Yap, 2013, p.6) Attempts to use the 'flipped classroom' methodology at tertiary setting (McGivney-Burelle & Xue, 2013) should also be reconsidered since Leader's observations about top quality student's inability or aversion to reading must have some truth in them.

On writing mathematics and the understanding of propositional logic (or syntactic understanding as conceptualised by Weber & Alcock (2004)), Leader explains the Cambridge approach which he admits works only because Cambridge can pay for the labour intensive pedagogy: "Cambridge is very lucky to have what we call the supervision system. So the students go to lectures, two hundred of them. … Lecturer gives out a problem sheet, sheet of exercises. The students will do the exercises, then once a week the students will have a supervision … that means two students to one professor, for an hour, each course. So, it's very intensive. And during that, the supervisor looks at the student's work, reads it carefully … and says … "This is badly written. Here's how you can improve it." So it's done by talking to the students. But that's a luxury tutorial system. … I don't believe in a separate expository writing class. I much rather they wrote the actual maths as how they learnt it. … So my personal feeling is that teaching students standard logic is a waste of time. If you try to teach students how to negate a statement "All pigs can fly." You teach them that then negation says, "Some pigs can't fly." So they get very good at that but still when they get to maths, they flip back and do things directly, in their own wrong way. So I think the best place to teach elementary logic is during the supervision, when they first have a maths theorem to prove, like "A implies B". And they say "Suppose A isn't true. Then B isn't true and I'm done." And then you correct it. I quite firmly believe that teaching logic separately is useless as they never learn to apply it in actual maths contexts." (Yap, 2013, p.7)

If the best students have such a hard time with learning mathematics if left to self-directed reading and writing mathematics, and the Cambridge solution involves close (and expensive) supervision, what pedagogical approaches would avail to the ordinary student in the ordinary university?

A curricular approach to teaching undergraduate mathematics

Alcock and Simpson note that students take a number of years for "development from an action through a process to an object conception before they begin to use the concept at university … [but] at the university level, a similar development is necessary, but a much shorter time period is available." (2009, p.22) Herein, perhaps, lies the difficulty that makes any well-intentioned pedagogy at university level flounder. There is not enough time. In addition, actions tend to be piecemeal from well-intentioned (or enlightened) lecturers and only implemented in their own courses. There is not enough time to teach reading and writing, understanding and construction of proofs, when the knowledge content needs to be covered.

A possible solution to the conundrum above would seem to be a curriculum review that rightly involves all who are teaching the curriculum. Tyler (1949) proposed a basic model of curriculum design that apparently is not utilized by most university faculty. In brief, Tyler's model requires first that the objectives of the curriculum be placed in a matrix with the modules of the programme so that the design can ascertain which cell in the table will be activated, i.e., which module can be used to attain the objective. The design requires the selection/development of learning experiences to achieve the objectives within the module.

The assessment and its modes of whether the objectives are achieved are also decided at the design stage.

The Mathematics and Mathematics Education department of the National Institute of Education has decided to use Tyler's model for the curriculum review and design of its undergraduate mathematics programme to take effect in 2016. The belief is that important process skills such as reading, writing and problem solving (Toh et al., 2014; Ho et al., 2014) can be thoughtfully spaced out over the four-year curriculum and a lecturer will know from the start what s/he has to emphasise and assess, and what has been done before his/her course, and what will follow further down the line. For example, reading can be spaced out as follows: reading of definitions (Year 1), reading of a short proof (Year 2), reading of definitions and proofs before a lecture (Year 3), reading of journal papers for honours dissertation (Year 4).

The feedback from faculty as they engage in curriculum review and design will be of great interest as such a collaboration is rare. We hope to be able to report in more detail on the progress of the review in December 2015.

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2. MATHEMATICS FOR MATH MAJORS

How do undergraduates read mathematical texts? An eye-movement study

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This paper reports on an eye-movement study of undergraduate mathematical reading behaviours. The eye movements of 38 undergraduate students were recorded as they read a multi-page textbook section on graph theory; participants then took a short comprehension test. This abstract reports basic results showing that neither reading time nor processing effort – measured via mean fixation durations – predicted comprehension test performance: students who read for longer or tried harder did not necessarily learn more. The conference report will include more detailed analysis of participants' eye movements: it will explore their relative attention to different parts of the text and the extent to which they shift their attention back and forth during learning, and will analyse the extent to which these behaviours differ across more and less effective learners.

Introduction

Undergraduate mathematics students are expected to learn in part by reading lecture notes and textbooks. But do they read effectively? Research shows that perhaps they do not. Interview studies indicate that when reading textbook passages, students tend to respond unhelpfully when facing confusion (Shepherd & van de Sande, 2014); eye-movement studies indicate that when reading a single purported proof, students tend to make less effort than mathematicians to study logical relationships between its claims (Inglis & Alcock, 2012).

This report will extend work of both types by reporting a study in which 38 undergraduate mathematics students read an extended graph theory textbook passage while their eye movements were recorded, then took a short comprehension test. It will report descriptive statistics showing dramatic variation in students' reading times and comprehension test scores, and analyses of differences in reading behaviours of more and less successful students, including their attempts to link different parts of the text, their relative attention to different representation types, and their relative attention to definitions, theorems, proofs and examples.

Theoretical Background

There has been increasing interest in recent years in undergraduates' mathematical reading behaviours and their consequences for comprehension. This has arisen in part because many lecture-based learning situations demand that students learn from written mathematics (Weber & Mejía-Ramos, 2014), and in part because researchers recognised that earlier work on proof had tended to focus on proof construction rather than on other issues such as comprehension (Mejía-Ramos & Inglis, 2009). Mathematicians have argued that comprehension tests and other activities related to proof evaluation can and should be used as a

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way to support critical engagement with complex mathematical arguments (Conradie & Frith, 2000; Kasman, 2006), and mathematics educators have done theoretical and empirical work in developing proof comprehension tests (Mejía-Ramos, Fuller, Weber, Rhoads & Samkoff, 2012).

Empirical study of broader mathematical reading behaviours nevertheless remains in its infancy, although studies of two types are contributing in different ways. First, interview studies indicate that when reading textbook passages, students tend to respond unhelpfully to confusion: compared with more mathematically experienced readers, they can be inattentive to details, insensitive to confusion or error, and less likely to seek resolution via careful re-reading (Shepherd & van de Sande, 2014). Such observations provide insight into suboptimal reading behaviours, but interview studies are necessarily subject to issues of reactivity (Russo, Johnson & Stephens, 1989): reporting aloud while learning can be expected to influence the behaviours under study.

Second, eye-movement studies indicate that compared with mathematicians, undergraduates attend less to the words of purported proofs and less to the logical relationships between their claims (Inglis & Alcock, 2012). Related work has demonstrated that self-explanation training can improve both attention to logical relationships and consequent comprehension (Hodds, Alcock & Inglis, 2014), but eye-movement work in this area has so far been restricted to the study of single proofs. It is thus limited in external validity: when studying lecture notes or textbooks, students need to understand extended passages of mathematical information; single proofs form part of such passages, but a student need not restrict their attention in this way.

The exploratory study reported here takes a step toward bringing together these approaches, studying eye movements of undergraduate mathematics students as they read an extended passage from graph theory text.

Method

The textbook section used was taken from the introductory chapter of the open-source textbook *Algorithmic Graph Theory* (Joyner, Nguyen & Phillips, 2011). Graph theory was considered appropriate because it requires few prerequisites and it commonly involves both verbal and algebraic arguments and diagrams. The first part of the chapter was formatted for eye tracking, with a standard font size but larger than usual spaces between lines; one definition and one diagram were repeated where this resulted in their being more separated from related content than they were in the book, and references to computer representations of graphs were removed. The resulting file took up 16 screens and included introductory material on vertices, edges, orders and sizes of graphs, adjacency and degree of vertices, regular graphs, subgraphs, walks, trails and paths, and connected, complete and cycle graphs. It contained several definitions, two sets of worked problems, two theorems with short proofs, one proposition with a lengthier proof, several diagrams, and passages of explanatory text.

A comprehension test was designed based on problems from the end of the chapter; because the number of questions on the included content was small, these were augmented with questions from a local graph theory course. Questions included multiple-choice items on basic definitions, drawing a graph and answering questions about its properties, and proving unseen results. The maximum score was 20.

Participants were mathematics students who had not taken a course in graph theory; each took part individually in exchange for a £6 inconvenience allowance. Participants were informed about the study's purpose and told that after the reading phase they would be asked to answer some questions without access to the textbook section. The eye-tracker was calibrated in each case, then participants read at their own pace. When they had finished, they were given 15 minutes to attempt the comprehension test and were asked to report their scores in earlier core mathematics courses; from these we constructed a prior performance measure. Forty students participated, and eye-movement data from 38 was of sufficient quality for analysis.

Results

Basic descriptive results are reported here; more detailed analyses of differences in reading behaviours are summarised and will be reported in detail at the conference.

Participants' prior performance scores ranged from 38% to 91% with a mean of 64.5%, meaning that they were representative of the student body as a whole (UK universities typically require 40% to pass a course and 70% for a first-class degree). Comprehension test scores ranged from 1 to 19 out of 20 with a mean of 9.68, and showed a moderate correlation with prior performance (r = .34, p = .036). This is unsurprising: one would expect mathematically stronger students to do better in both, but short-term learning from a single text and longer-term learning from more materials obviously demand different skills.

Total reading durations varied widely, ranging from 13 to 35 minutes with a mean of 20.5 minutes; nevertheless they did not significantly correlation with comprehension test score (r = -.08, p = .645). This is striking: if longer study time does not reliably need to greater learning, then some students must use their reading time considerably more effectively. A similar result was found for mean fixation durations, where longer fixation durations are associated with greater processing effort (Rayner, 1998). Mean fixation durations were not significantly correlated with comprehension test scores (r = -.08, p = .624). Thus neither time nor effort predicted learning outcomes in the obvious way.

To investigate more localised differences in reading behaviour we divided the text into areas of interest (AOIs) (Tobii Technology, 2010), one for each title, quote, definition, example, theorem, proof, diagram, problem, and worked solution. To assess participants' attempts to link different parts of the text, we analysed participants' total *visit counts*, where a visit is a set of consecutive fixations in an AOI. When controlling for reading time there was no significant difference between higher- and lower-performing students on this measure. We note, however, that visit count is only a proxy for shifts of attention – studies of single proofs have considered between-line saccades (Inglis & Alcock, 2012) – and it is not obvious how best to study this aspect of reading behaviour for extended passages of text.

To assess participants' distribution of attention across different types of text we calculated the proportions of their reading times spent on these types. Students who performed better in the comprehension test paid less attention to examples and more to definitions and theorems, a result consistent with long-established arguments about the need for students to understand the importance of definitions in mathematics (Vinner, 1991). We note, however, that effect sizes were small, and that there remain numerous questions about how best to study and understand differences in mathematical reading behaviours. We will report on the details of these analyses and discuss the methodological issues further at the conference.

Discussion

This study was designed to explore undergraduates' mathematical reading behaviours during study of an extended textbook passage. Eye-tracking allows us to do this in an unobtrusive way because it provides behavioural measures without requiring participants to articulate their thoughts aloud. Of course, it has limitations: one commonly-offered critique is that eye-movement analyses require students to read on a screen, and that this is different from reading mathematics on paper with a pen in hand. While this is indisputable, reading on a screen is a common activity in contemporary education: both students and mathematicians routinely access information in this way. More importantly, it cannot account for between-participant differences: all participants in the reported study were in the same position. Nevertheless, we agree that external validity remains an issue, and future research might well look to use mobile recording methods to study mathematical reading 'in the wild' (cf. Savic, 2015).

In the meantime, our early analyses indicate that obvious variations in reading duration and effort do not account for differences in learning effectiveness, and that explanations for this must therefore reside in other aspects of reading behaviour. At the conference we will report in detail on participants' relative attention to different aspects of the text, analyse the extent to which this differs across more and less effective learners, and discuss follow-up research questions that would be open to investigation using a variety of methodological approaches.

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Undergraduates learning of programming for simulation and investigation of mathematics concepts and real-world modeling

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In a position paper for the European Commission's contributions to European Research, the European Mathematical Society (2011) recently stated that: "Together with theory and experimentation, a third pillar of scientific inquiry of complex systems has emerged in the form of a combination of modeling, simulation, optimization and visualization." (p.2). How should university mathematics education respond to this newly identified 'third pillar of scientific inquiry' for complex systems? In this extended abstract, I briefly discuss an implementation at Brock University (Canada) since 2001 of a sequence of three undergraduate, constructionist mathematics courses that aims at supporting students' development of proficiency in this third pillar.

Introduction

At Brock University (Canada), all undergraduate mathematics majors and future mathematics teachers learn, as part of a *sequence of three core* mathematics courses[1], to design, program, and use interactive computer environments, that we have called *exploratory objects* (EO), for simulation and investigation of mathematics concepts, conjectures, and real-world modelling (Muller et al., 2009). These project-based MICA courses, an acronym for *Mathematics Integrated with Computers and Applications* (Ralph, 2001), can be seen to support students' development of proficiency in the third pillar of scientific inquiry of complex systems as described by the *EMS* (2011) (Buteau et al., 2016); indeed, an official departmental document that led to adopt these courses in 2001, states (Buteau et al., 2016, [p.3]):

> *In dealing with such problems [from pure and applied mathematics that require experimental and heuristic approaches], students [in MICA courses] will be expected to develop their own strategies and make their own choices about the best combination of mathematics and computing required in finding solutions… [The] goal is to help students build a portfolio of techniques which they are confident in applying to a diverse range of mathematical problems that may or may not have exact solution.*

In the following, I briefly discuss the MICA classroom implementation and student experiences, and end with a mention of forthcoming research.

The classroom implementation: an integrated model

Each MICA course is designed around project assignments that count for about 75% of students' final grades. For the three MICA courses, there are in total 14 individual projects, eleven of which on topics assigned by the instructor and the other three at the end of each

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[1] Recent changes in the undergraduate program will affect these courses mostly starting in 2016-17. This paper discusses the implementation that has been taking place from 2001 to 2015.

course on a topic of the student's own choice (Buteau et al., 2014a). As an example of an instructor's selected topic, students enrolled in 3rd MICA course in 2012 were required to program cellular automata processes to simulate real-time epidemic spreads, investigate their evolution and effect of inoculation, and explore their approximate related costs (Buteau et al., 2014a). Figure 1 provides two interface frames of a student's EO. We argue that this work exemplifies an instance of students (guided to) engage in the third pillar of scientific inquiry.

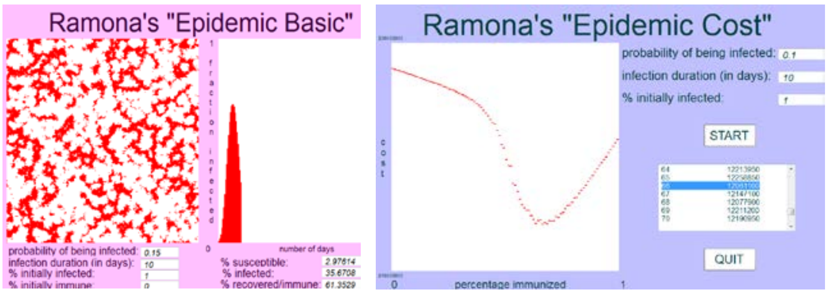

Figure 1. Screenshots of a second-year student's, Ramona, assigned EO project about simulating and investigating epidemics and their related costs (Buteau et al., 2014a).

The pedagogical MICA course design chosen by the department involves a course format of 2 weekly hours of lecture (mathematics content) together with a two hour computer lab session (programming-based mathematics tasks). It also involves an integrated teaching model wherein students learn computer programming within these mathematics courses in conjunction with mathematical concepts of increasing complexity (most students enrol in MICA courses with no programming background). In particular, this means that the MICA I course has been carefully designed to support the students' instrumental genesis (Rabardel 1995/2002) of programming needed for engagement in the third pillar (Buteau & Muller, 2014).

In addition, the pedagogy adopted in MICA courses requires a different teaching paradigm than the common university model of the delivery information followed by the students working on instructor developed problems (Muller et al., 2009). The role of the instructor becomes one of a facilitator. For example, as part of becoming proficient in the third pillar, we argue that one should develop the ability to conjecture, and identify and state mathematics problems. For example in MICA I lectures, some classroom time is devoted to activities wherein students regroup to develop conjectures on a topic (e.g. prime numbers), followed by a classroom discussion on the precision, importance, and relations among conjectures and then by a classroom discussion on the relevance of programming technology to investigate them (Muller et al., 2009). This experimental mathematics aspect of the MICA courses is also reflected in all EO projects (Marshall & Buteau, 2014).

Overall, the pedagogical paradigm adopted in the MICA courses follows that of a constructionism approach (Papert, 1991); for example, Papert (1980) stated that in a microworld situation, "*the relationship of the teacher to learner is very different: the teacher introduces the learner to the microworld in which discoveries will be made, rather than to the discovery itself*" (p. 209). Similarly, an instructor in MICA courses introduces during lectures mathematics concepts that ground a mathematical context for (motivating) students to investigate them, in lab sessions or through their EO assignments: the students create (i.e., design, program, etc.) a computer environment with interface which they use and amend as required to investigate the mathematical concept, conjecture or real world application. In short, the MICA instructor presents a situation to solicit students to think like mathematicians engaging in the third pillar of scientific inquiry rather than to teach them about mathematics.

The students' learning experiences

When creating and using an EO for their investigation of a mathematics concept, conjecture, or real-world situation, students in MICA courses (are guided to) engage in the third pillar of scientific inquiry (Buteau et al., 2016). Diagram 1 summarizes a student's engagement during his/her EO work (Buteau & Muller, 2010) which, we argue, reflects well the work of a mathematician engaging in the third pillar. For students, the mathematical work in each EO involves mainly two aspects: the programming of (known) mathematics concepts and the investigation using the EO interface (according to an experimental mathematics approach).

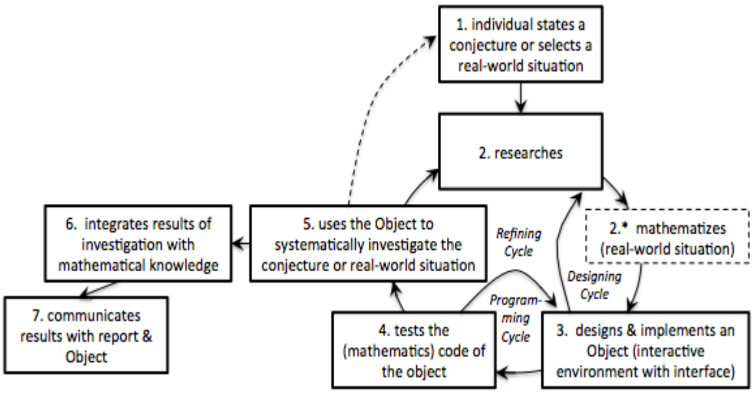

Diagram 1. Development process model of a student creating a computer environment for a mathematical investigation or application (modeling or simulation) (Buteau & Muller, 2010; Marshall & Buteau, 2014).

Also, a recent preliminary empirical study (student survey, N = 56) provides evidence that students, who have completed the MICA courses, view that they have engaged in the third pillar of scientific inquiry (Buteau et al., 2015). For example, a student describes the MICA courses in the following manner:

> *The courses teach you how to use an interactive programming environment ... and allow you to use it to investigate different mathematical theorems and concepts. It is very effective because it allows you to make your own program to be able to see how this concept works, and play around with it to reach a further understanding of the concept;*

and a graduate from MICA courses, also a teaching assistant in MICA II course, describes:

> *Overall, these courses are meant to provide students with both the tools and the mindset to tackle a wide variety of mathematical problems efficiently, through the use of modern computer softwares. (Buteau et al., 2015, p.147)*

The empirical study also suggests that students view developing, as they progress through their MICA courses, 15 key competencies in relation to the third pillar; for example i) to self-motivate to learn/do mathematics; ii) to engage in the process of mathematics research; and iii) to closely reflect on problems (Buteau et al., 2014b). In fact, most of these 15 competencies align well with results from research on programming-based, constructionist approach to mathematics learning such as Wilensky's (1995) (Marshall et al., 2014).

Furthermore, our recent case study that closely examined a student's overall 14 EO work (Buteau et al., 2016) suggests that students may view experiencing a sense of empowerment in terms of creation and validation of mathematical knowledge; e.g., a student writes: "*I have developed [the knowledge] about how to creatively think about a problem for ways that it could be modelled.*" This is reflected in the student's following quote suggesting that he views having become proficient, i.e., confident and skilled, in engaging in the third pillar:

> *…The possibilities are only limited to the creativity of the mathematician making the models. I think the major skill I will take with me from MICA courses is the ability to create, analyse and explore dynamical systems and make the connections between them and the real world. (Buteau et al., 2016, [p.20])*

Future research

The case study mentioned above is preliminary to a forthcoming comprehensive study on MICA students' appropriation of programming as an instrument for engagement in the third pillar, framed mainly in the context of an instrumental approach (Rabardel 1995/2002).

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Duality between formalism and meaning in the learning of linear algebra

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In the French tradition of Bourbaki, the theory of vector spaces is usually presented in a very formal setting, which causes severe difficulties to many students. The aim of this talk is to analyze the underlying reasons of these difficulties and to suggest some ways to make the first teaching of the theory of vector spaces less inefficient for many students. From a mathematical analysis with a historical perspective, we analyze the teaching and the apprehension of vector space theory in a new approach. I particular, we w show that mistakes made by many students can be interpreted as a result of a lack of connection between the new formal concepts and their conceptions previously acquired in more restricted, but more intuitively based areas.

Linear algebra: a first encounter with formalism

Linear algebra is universally recognized as a very important subject in the mathematics curriculum in many universities. Usually linear algebra represents the first contact with such a "modern" axiomatic approach. However, since the 90s in many universities, the teaching of linear algebra became less formal and it is often preceded by a preparatory course in Cartesian geometry or/and by a course in logic and set theory. Yet, in secondary school, students still learn the bases of vector geometry and the solving of systems of linear equations by Gaussian elimination. Therefore, they have some knowledge on which the teaching of linear algebra can be based. For the moment, the idea of teaching students the axiomatic elementary theory of vector spaces within the first two years of science university has not be questioned seriously, and the teaching of linear algebra, in most countries, remains quite formal.

In the 80s already, A. Robert and J. Robinet (1989) showed that the main criticisms made by students toward linear algebra concern the use of formalism, the overwhelming amount of new definitions and the lack of connection with what they already know in mathematics. It is quite clear that many students have the feeling of landing on a new planet and are not able to find their way in this new world. On the other hand, teachers usually complain of their students' erratic use of the basic tools of logic or set theory, and have no skills in elementary Cartesian geometry and consequently cannot use intuition to built geometrical representation of the basic concepts of the theory of vector spaces. In my doctorate (Dorier 1990), I tested with statistical tools the correlation between the difficulties with the use of the formal definition of linear independence and the difficulties with the use of the mathematical implication in different contexts. The results showed clearly that no systematic correlation could be made. This means that students' difficulties with the formal aspect of the theory of vector space are not just a general problem with formalism and logic but mostly a difficulty of understanding the specific use of formalism and logic in the theory of vector

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spaces and the interpretation of the formal concepts in relation with more intuitive contexts like geometry or systems of linear equations, in which they historically emerged. I will analyze this point in more details in this paper on the specific point of linear independence.

Linear independence

Liner independence is one of the most basic notions in vector space theory. Yet, even if students can be easily trained to solve standard questions like "is this set of vectors independent or not?" in various contexts, the use of these notions in less straightforward situations may be much less easy. On the other hand, the historical genesis of this concept is maybe less trivial as could be imagined. I will first give some accounts on students' difficulties, I will then give a brief account of the historical evolution and finally draw out some conclusion on the basis of a coordinated epistemological synthesis of the first two points.

Students' difficulties

In a standard linear algebra course, students are trained to check whether a set of *n*-tuples, equations, polynomials or functions are linearly independent. This technical part of the learning is often quite easy going. However it does not mean that the same students are able to use the concept of linear independence in more formal contexts.

For instance, let us consider the following two conjectures:

1. Let U, V and W be three vectors in $\boldsymbol{R}^3$, and f a linear operator in $\boldsymbol{R}^3$, if U, V and W are independent, $f(U), f(V)$ and $f(W)$ are independent.
2. Let U, V and W be three vectors in $\boldsymbol{R}^3$, and f a linear operator in $\boldsymbol{R}^3$, if $f(U), f(V)$ and $f(W)$ are independent, U, V and W are independent.

When asked whether each of these conjectures is true or false. Many students have the feeling that the first is true and the second is false… On the basis of this first impression, beginners usually try to use the formal definition, without first checking on concrete examples. They try different combinations with the hypotheses and the conclusions, and very often give a proof that shows some difficulty in the use of the formal definition of linear independence, but my analysis led to show that this is not just a lack of ability with the use of logic and implication. (Dorier, 1997, pp. 116-121, Dorier 1998 or Dorier, 2000, pp. 95-103). If a certain level of ability in logic is necessary to understand the formalism of the theory of vector spaces, general knowledge, rather than specific competence is needed. Furthermore, if some difficulties in linear algebra are due to formalism, they are specific to linear algebra and have to be overcome essentially in this context.

On the other hand, some teachers may argue that, in general, students have many difficulties with proof and rigor. Several experiments that we have made with students showed that if they have connected the formal concepts with more intuitive conceptions, they are in fact able to build very rigorous proves. This implies not only giving examples but also to show how all these examples are connected and what is the role of the formal concepts with regard to the mathematical activity involved.

Historical background

Formalized axiomatic theory of vector space dates only from the early 20th century (see Dorier 1993, 1995a, 2000 – 1st part). The first time the question of dependence was discussed is in a text published by Leonhard Euler (1707-1783) in 1750 about what is known as Cramer's paradox. In this text, Euler does not use the term "dependence", he says that an equation is "contained" or "comprised" in others. His idea is that this equation does not bring more new restriction on the unknown than the others. There his conception of "dependence" is not quite the same and I have use the term "inclusive dependence" to distinguish it from the conception in terms of linear combination being zero. Mathematically speaking, the linear dependence between n equations in n unknowns is equivalent to the fact that the system will not have a unique solution. However, the two properties correspond to two different conceptions of dependence, and the inclusive conception is natural in the context in which Euler and all the mathematicians of his time were working, that is to say with regard to the solving of linear equations, and not the study of equations as objects on their own. Yet, there is a difficulty for further development regarding the concept of rank. 1750 is also the year Gabriel Cramer (1704-1752) published the treatise that introduced the use of determinants which was to dominate the study of linear equations until the first quarter of the 20th century. In this context, dependence was characterized by the vanishing of the determinant. The notion of linear dependence, now basic in modern linear algebra, did not appear in its modern form until 1875. Ferdinand Georg Frobenius (1849-1917) introduced it, pointing out the similarity with the same notion for n-tuples. He was therefore able to consider linear equations and n-tuples as identical objects with regard to linearity. This simple fact may not seem very relevant but it happened to be one of the main steps toward a complete understanding of the concept of rank. Indeed in the same text, Frobenius was able not only to define what we would call a basis of solutions but he also associated a system of equations to such a basis (each n-tuple is transformed into an equation). Then he showed that any basis of solutions of this associated system has an associated system with the same set of solutions as the initial system. This first result on duality in finite-dimensional vector spaces showed the double level of invariance connected to rank both for the system and for the set of solutions. Moreover, Frobenius' approach allowed a system to be seen as an element of a class of equivalent systems having the same set of solutions: a fundamental step toward the representation of sub-spaces by equations.

This brief summary of over a century of history shows how adopting a formal definition (here of linear dependence and independence) may be a fundamental step in the construction of a theory, and is therefore an essential intrinsic constituent of this theory.

Conclusions

Students must be aware of the unifying and generalizing nature of the formal concept. In our research, we used what we called the *meta lever* to build teaching situations leading students to reflect on the nature of the concepts with explicit reference to their previous knowledge (Dorier 1995b, 1997 and 2000 (II.4) and Dorier et al. 2000). In this approach, the historical analysis is a source of inspiration as well as a means of control. Nevertheless, these activities must not only involve a lecture by the teacher, nor a reconstruction of the historical development, but take into account the specific constraints of the teaching context, to reconstruct an evolution of the concepts with consistent meaning.

For instance, with regard to linear (in)dependence, based on students' good practice of Gaussian elimination for solving systems of linear equations, we can make them reflect on this technique not only as a tool but also as a means to investigate the properties of the systems of linear equations. This does not conform to the historical development, as the study of linear equations was historically mostly held within the theory of determinants. Yet, Gaussian elimination is a much less technical tool and a better way for showing the connection between inclusive dependence and linear dependence as identical equations (in the case when the equations are dependent) are obtained by successive linear combinations of the initial equations. Moreover, this is a context in which such question as "what is the relation between the size of the set of solutions of a homogeneous system and the number of relations of dependence between the equations?" can be investigated with the students as a first intuitive approach for the concept of rank. Rogalski has experimented with teaching sequences illustrating these ideas (Rogalski 1991, Dorier 1997 and 2000, II.3). In these experiments, we have also proposed a different way to introduce the concepts of linear dependence and independence that introducing the formal theory after having made as many connections as possible with previous knowledge and conceptions in order to build better intuitive foundations.

The theory of vector spaces is a unifying and generalizing theory, in the sense that, historically, not only did it allow solving new problems in mathematics, but it essentially unified tools, methods and results from various backgrounds in a very general approach. Thus its formalism is a constituent of its nature. Yet all the problems our students may solve with this theory could be solved with less sophisticated tools which they have already learnt (or at least are supposed to have learnt). Therefore the gains of this unification and generalization have to be understood by them, if we want them to accept this formalism and to use the theory correctly.

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Use of letters in mathematics at university level teachers' practices and students' difficulties

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In this communication, we will question the possible impact on students learning of the rather common practice at tertiary level of a floating use of letters in proof and proving. We first provide a brief logical background concerning the logical status of letter relying on Copi's natural deduction. We then provide an example out of a textbook address to undergraduates enlightening what we mean by "floating use of letters". Finally, we will put this practice in relationship with students' difficulties.

Introduction

In mathematics education, at tertiary level, there is an increasing use of letters with various logical status: variable, either free or in the scope of a quantifier (universal of existential); singular element; generic element. Taking in account the logical status of a letter in a proof is often a clue aspect in proof and proving. Nevertheless, this remains often implicit in mathematical texts address to students, in particular when teachers consider it is not dangerous (Durand-Guerrier and Arsac, 2005). In this communication, we will question the possible impact on students learning of the rather common practice at tertiary level of a floating use of letters in proof and proving. We will first provide a brief logical background concerning the logical status of letter relying on Copi's natural deduction, considering *introduction* and *elimination* of quantifiers as a mean to explicit this logical status. We will then provide an example a floating use of letters in mathematical a textbook on advanced calculus. Finally, we will suggest that this practice could reinforce some students' difficulties.

Relationship between quantification and logical status of letters

In line with several authors (i.e. Dubinsky & Yparaki, 2000, Selden & Selden, 1995, Epp 2004) we claim that quantification matters are playing an essential role in mathematical activity and conceptualization. In this respect, we consider that first order logic developed in the late nineteenth and early twentieth centuries by logicians such as Frege, Russell, Wittgenstein, and Tarski, is a relevant epistemological reference for analyzing mathematical discourse, as well written that oral. Among the relevant aspects for a didactic perspective we focus in this paper on the dialectics between syntax and semantic, form and content, logical validity and truth in an interpretation, theoretically founded by Tarski (1933) and popularized by Quine (1950). These dialectics provide a relevant background for analyzing mathematical activity, including but not restricted to proof and proving in mathematics (Durand-Guerrier, 2008, Durand-Guerrier & al. 2012). A main feature of first order logic is to analyze the statement in object, properties, relationships and quantifiers. As a consequence the is-

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urn:nbn:de:hebis:34-2016041950121

sue of the logical status of letter is crucial: a letter may represent a *singular object* (a constant) or a *generic element* (an element consider only as a member of a given set or subset); a letter may also have the logical status of *free variable* in an atomic formula, that does not represent anything but is subject to assignation, or *a bounded variable* in the scope of a quantifier (a particular case of *mute letter*). Possibilities to deal with letters in mathematical activity are constrained by their logical status. However, in mathematics and in undergraduate mathematics education, it is very common to let implicit the logical status of letters involved in an activity, a reasoning or a proof, in particular via the spread use of implicit quantification on universal conditional statements; we have shown that this is likely to induce difficulties for novice university students (Durand-Guerrier et Arsac, 2005). A pending question in mathematic education concerns the balance between rigor requirements and pragmatic aspect of mathematical reasoning and heuristic. The systems of natural deduction have been developed to provide a tool for analyzing effective mathematical reasoning. In our work, we use the system developed by Copi (1954) as a tool for analyzing the logical status of letters in proofs, reasoning and more generally in mathematical discourse in a didactic perspective.

Logical status of letter in the light of Copi's natural deduction

The main interest of natural deduction in a didactic perspective relies in the rule for elimination and introduction of logical connectors and quantifiers (Durand-Guerrier et Arsac, 2015). As we have said, we pay interest in our research on first order logic, which deals not only with propositions, but also predicates, variables and quantifiers. This needs to introduce rules for introduction and elimination of quantifiers. In Copi's system, the rules involved an implicit generic domain of interpretation, and so they appear as formalization of the dialectic between *formal statements* in Predicate calculus and *interpretation* in a given *universe of discourse*. There are four rules concerning quantifiers; two of them are relying on logically invalid quantificational schema so that restrictions are required to insure validity. We present below the version of Copi (1954, second edition 1965).

The first rule of inference concerns elimination of the universal quantifier; it is called Universal Instantiation (U.I.) and states that: "(…) any substitution instance of a propositional function can validly be inferred from the universal quantification" (Copi, 1965, p.50). This rule relies on the valid schema "$(x)\Phi(x) \Rightarrow \Phi(y)$" and expresses that "What is true for all is true for any". The second rule is the dual of the first one and concerns the introduction of the universal quantifier; it is called Universal Generalization (U.G.) and states that: "(…) the universal quantification of a proposition can validly be inferred from a substitution instance with respect with the symbol y." (ibid., p.51). This rule relies on the invalid schema $\Phi(y) \Rightarrow (x)\Phi(x)$. For this reason, it necessitates a restriction; you must be sure that no assumption other than the property expressed by Φ has been done. Obviously, it is build "by analogy with a fairly standard mathematical practice" (ibid., p.50). The third rule concerns the introduction of the existential quantifier; it is called Existential Generalization (E.G.) and states that: "(…) the existential quantification of a propositional function can be validly inferred from any substitution instance of that propositional function." (ibid., p.52). The fourth rule concerns the elimination of the existential quantifier; its is call Existential Instantiation (E.I.) and states that: "(…) from the existential quantification of a propositional function we may validly infer the truth of its substitution instance with respect to an individual constant which has no prior occurrence in that context." (Ibid. p.52). As the second rule, this one relies on an invalid schema. This fourth rule is the more delicate to use, and necessitates a global control of the proof or of the argument. Indeed, as we have shown in Durand-Guerrier et Arsac (2005), a control step by step is not enough to track possible invalid deduction.

Floating use of letter in mathematical text addressed to undergraduates

Although it seems indubitable that a correct use of letters in mathematics is a clue competence at university, we can observe in many mathematical texts addressed to students a floating use of letters in the following sense: in a text unit, it is not seldom that the logical status of a same letter is changing along the text. For example, the aim is to prove a universal statement in which the bounded variable is named x. The first thing is to introduce a generic element (U.I.), most often, it is also named x. In other cases, at some moment a singular element named x is proved to satisfy a given property P; then the conclusion is "there exists x so that $P(x)$" (E.G.). This practice is opposite with the choices made by Copi who changes the category of letters according with their logical status (x, y, z, .. for variables; a, b, c... for generic or singular elements). We find such floating use of letters in numerous proofs in textbooks, as in the following example (Figure 1):

For another example, let $f(x) = 1/x$. We shall show that f is continuous on the open interval $0 < x < 1$ but is not uniformly continuous there. We first write

$$|f(x) - f(x_0)| = \left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x_0 - x|}{xx_0}$$

To prove continuity at x_0, which may be any point with $0 < x_0 < 1$, we wish to make $|f(x) - f(x_0)|$ small by controlling $|x - x_0|$. If we decide to consider only numbers δ obeying $\delta < x_0/2$, then any point x such that $|x - x_0| < \delta$ must also satisfy $x > x_0/2$, and $xx_0 > x_0^2/2$. Thus, for such x, $|f(x) - f(x_0)| < \delta/xx_0 < 2\delta/x_0^2$. Given $\epsilon > 0$, we can ensure that $|f(x) - f(x_0)| < \epsilon$ by taking δ so that $\delta \leq (x_0^2/2)\epsilon$. Thus, f is continuous at each point x_0 with $0 < x_0 < 1$. If f were uniformly continuous there, then a number $\delta > 0$ could be so chosen that $|f(x) - f(x')| < 1$ for every pair of points x and x' between 0 and 1 with $|x - x'| \leq \delta$. To show that this is not the case, we consider the special pairs, $x = 1/n$ and $x' = \delta + 1/n$. For these, we have $|x - x'| = \delta$ and

$$|f(x) - f(x')| = \left|n - \frac{1}{\delta + 1/n}\right| = \frac{n\delta}{\delta + 1/n}$$

No matter how small δ is, n can be chosen so that this difference is larger than 1; for example, any n bigger than both $1/\delta$ and 3 will suffice.

Figure 1: An example of proof with floating use of letter (Buck et Buck, 1965, p. 68).

In this proof, the letter x is first used as a free variable in the definition of function f; then x intervenes in an unusual notation for the open interval]0;1[. In the mathematical expression written on line 4, although x and x_0 have not been explicitly introduced, they seem to represent *generic elements*. This status of generic element for x_0 is confirmed in line 5, but nothing is said concerning x. On lines 7 and 8, the letter x intervenes in a sentence that could be formalized as a universal conditional statement in which x would be a *bounded variable*. On line 9, in the first sentence the letter x denotes a *generic element* satisfying "$|x - x_0| < \delta$". In the following sentence, "Given $\varepsilon > 0$, we can ensure that "$|f(x) - f(x_0)| < \varepsilon$ by taking δ so that $\delta \leq (x_0^2/2)\varepsilon$", there is no indication that the letter x is a *bounded variable* in the scope of an implicit universal quantifier, corresponding to a Universal Generalization on x. As a consequence, the difference of logical status between the generic element x_0, and the bounded variable x remains implicit, while in the next sentence on the property that f is not uniformly continuous, the universal quantification of both x and x' is explicit. Finally, x and x' are used *to label* the pairs $(1/n;\ \delta + 1/n)$ depending on n, without any indication of this dependence.

Although we have not made an extensive inquiry, we hypothesize that this example is representative of what we can find in mathematical textbooks addressed to undergraduates.

University students difficulties related to floating use of letters

Of course, for university teachers a floating use of letters is not problematic as they have semantic controls that prevent them from invalid mathematical arguments. Nevertheless, it is well known from history of mathematics that even prominent mathematicians have produced invalid proofs. Concerning students, in line with other authors (i.e. Selden and Selden, 1995) we have shown that uncertainty on the logical status of letter provoke difficulties in proof and proving in mathematics, in particular when multiquantified statements are involved (Durand-Guerrier et Arsac, 2005, Chellougui 2009). We will present during the conference recent results showing that this might be source of conceptual difficulties.

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When is a parabola not a parabola?

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We use the idea of sibling curves to visualize complex polynomials and their zeroes and see that a parabola is a singular case of a complex quadratic. Being four dimensional, it is problematic to visualize graphs and roots of polynomials with complex coefficients in spite of many attempts through centuries. Three dimensional sibling curves are introduced by restricting the domain of a complex function of a complex variable to those complex numbers that map onto real numbers, resulting in new representations of functions other than the well-known curves in the real plane that only depict part of a bigger whole. The expanded representation brings new insight into the understanding of complex polynomials. In the case of a complex quadratic we see that the sibling curves lie on a three-dimensional hyperbolic paraboloid.

Introduction

For centuries mathematicians have spent time and energy on how to visualize polynomials with complex coefficients and how to visualize complex zeroes of polynomials. Roots of a function have been visualized as the points of intersection with the "floor" (horizontal axis or plane). Since the graph of a complex function on a complex domain is four-dimensional, visualization is not evident. We have two planes, the Cartesian plane that can be used for showing the graph of a function and where its real roots lie and the Argand plane that gives a visual representation of complex numbers. Harding and Engelbrecht (2007a) followed the historical journey of finding roots of complex functions graphically but all attempts are somewhat artificial.

A new idea appeared in print in 1951, in an American secondary school textbook (Fehr, 1951). This approach was further developed by Harding and Engelbrecht (2007b). If we restrict the domain of f to those complex numbers that map onto real values, on this restricted domain the function can be represented in *three* dimensions with the domain in the horizontal plane (the complex plane) and the range along the vertical (real) axis. This led to the idea of *sibling curves,* which turn out to be a rich and useful way of visualizing zeroes of polynomials and other well-known functions as well as visualizing complex functions in *three* dimensions.

In particular, a polynomial f can be written in the form:

$$w = f(z) = f(x + iy) = g(x, y) + ih(x, y)$$

for some polynomials g and h. If f maps the complex number $x + iy$ onto a real number then $h(x, y) = 0$. Restricting the domain of the function f to these points in the xy −plane, the condition $h(x, y) = 0$ defines a curve(s) in the Argand plane (the horizontal plane). The function f with these curves as domain form the *sibling curves*.

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urn:nbn:de:hebis:34-2016041950121

Example. If $f(z) = z^2 + 2z + 2$ and z is a complex number $x + iy$ then

$$f(z) = (x^2 - y^2 + 2x + 2) + 2iy(x + 1)$$

So $f(z)$ is real in the plane $y = 0$ (the Cartesian plane) and in the plane $x = -1$ (a plane perpendicular to the Cartesian plane). In the real plane $y = 0$ the function values are given by

$$f(x) = x^2 + 2x + 2,\ x \in \mathbb{R},$$

which is the well-known parabola that we have always considered as the entire graph.

In the plane $x = -1$ perpendicular to the Cartesian plane, the function is $f(y) = -y^2 + 1$, $y \in \mathbb{R}$. So the parabola as we know it is one of twin curves, defined in planes perpendicular to each other. These twin curves are called the *sibling curves* of the function. The function has complex roots $z = 1 \pm i$, and visually it is evident that these roots are situated where the function $f(y)$ cuts the complex (horizontal) plane.

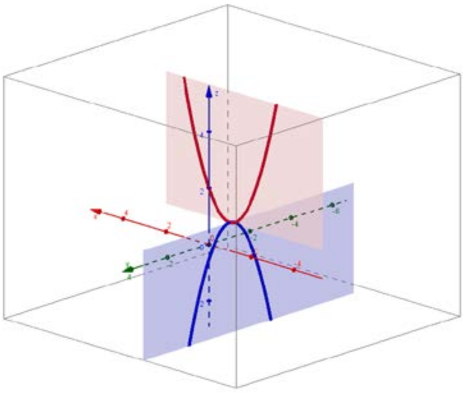


Figure 1: The quadratic function $f(z) = z^2 + 2z + 2$

Expanding on the idea outlined above, Harding and Engelbrecht (2007b) developed sibling curves of a number of well-known functions, including cubics, quartics, exponential, trigonometric and hyperbolic functions.

The work on sibling curves was given a more solid mathematical founding in (Engelbrecht, Fouche, Harding, & Wiggins, 2015) in which the main result proven is that a polynomial with degree n has n sibling curves. This theorem is also true for polynomials with complex coefficients.

Real and complex quadratics

For quadratic polynomials where the coefficients are real, Wiggins, Harding, & Engelbrecht, (submitted) proved that we get two *congruent* (a distance preserving bijection existing between the curves) sibling parabolas that always meet in one point, the turning point of the original parabola. From Figure 1 it appears that the other sibling curve is also a parabola. This is indeed the case (Wiggins, et al. (submitted)).

If the coefficients are complex numbers, Wiggins, et al. (submitted) proved that the two sibling curves intersect if and only if $\Delta = \frac{4ac-b^2}{4a}$ is a real number. So the sibling curves in this case do not always intersect and are not always parabolas.

Example. Consider $f(z) = (z-1)(z-i) = z^2 - (1+i)z + i$.

Let $z = x + iy$, then

$$f(z) = (x+iy)^2 - (1+i)(x+iy) + i = (x^2 - y^2 - x + y) + i(2xy - x - y + 1).$$

If $f(z)$ is real then $2xy - x - y + 1 = 0$ which means that the projection is hyperbolic.

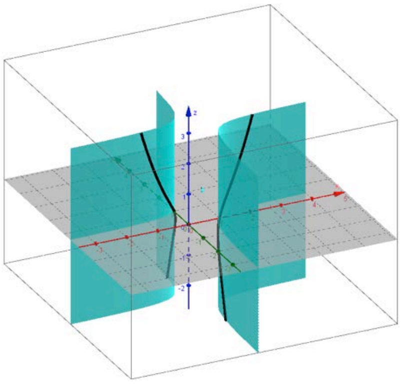


Figure 2: Sibling curves of $f(z) = (z-1)(z-i)$

So for complex quadratics, the sibling curves of these quadratics are not necessarily longer parabolas and the projection of these sibling curves on the complex domain is a hyperbola and not two perpendicular straight lines as was the case with real quadratic polynomials.

Hyperbolic paraboloid

For quadratic polynomials, if the sibling curves meet they are two planar parabolas, each sibling curve lying in its own plane. If the two sibling curves do not meet, the curves are not parabolas, but again congruent curves (Wiggins, et al., (submitted)).

Without loss of generality, we only need to consider the quadratic polynomial

$f(z) = z^2 + C$ for some complex number C. If $z = x + iy$, then

$$f(z) = x^2 - y^2 + Re(C) + i\big(2xy + Im(C)\big).$$

Hence the sibling curves always lie on a scaling, translation or rotation of the hyperbolic paraboloid $z = x^2 - y^2$ (saddle shaped surface or "Pringle").

In Figure 3 we see the typical scenario when the two sibling curves meet. The two parabolas meet in the saddle point of the hyperbolic paraboloid – both parabolas' turning point.

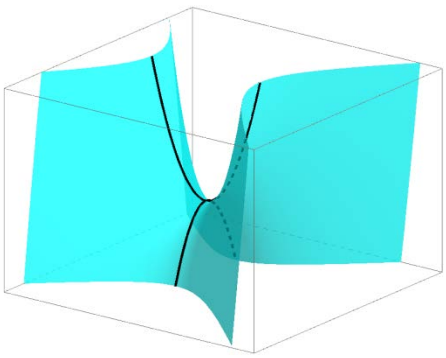
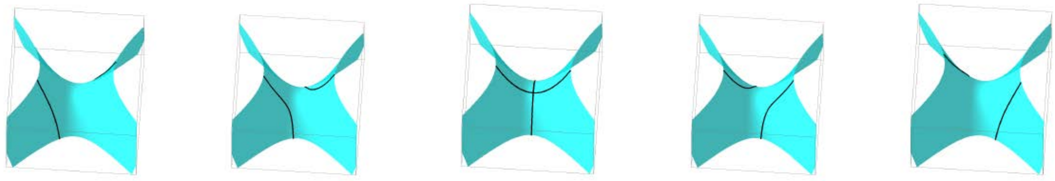


Figure 3: Sibling curves of $f(z) = z^2 - 1$ ***on a hyperbolic paraboloid.***

An animation of the sibling curves of the quadratic $f(z) = z^2 + 2z + (1 + ik)$ can be seen at https://cardanogroup.wordpress.com/. In Figure 4 we sketched snapshots of the sibling curves of this quadratic for various real values of k. Notice that when $k = 0$ the two sibling curves are parabolas that intersect in a point.



(a) $k = -2$ (b) $k = -1$ (c) $k = 0$ (d) $k = 1$ (e) $k = 2$

Figure 4: Snapshots of some sibling curves of $f(z) = z^2 + 2z + (1 + ik)$

Quite surprisingly we notice from this animation (or looking at the snapshots of the animation) that one half of the parabola joins up with another half of the other parabola to form new sibling curves when $k \neq 0$.

The general case

We considered a three-dimensional cut of the four-dimensional graph by only considering those complex numbers in the domain of which the function values are real. This is a special case since we do not see the complete four-dimensional object.

If, however, we let

$$f(z) = az^2 + bz + c = ke^{i\theta} \in \mathbb{C} \text{ then}$$
$$az^2e^{-i\theta} + bze^{-i\theta} + ce^{-i\theta} = k \in \mathbb{R}$$

which is exactly the case that was discussed.

This fresh insight implies that by only considering the case where the function values are real we place no limitation on the images created representing all quadratics – nothing new will be seen if we include complex function values in the range instead of being limited to real function values.

In conclusion

The findings in this paper show that parabolas are singular cases of a general quadratic. We now have a new perspective on the shape of a parabola as traditionally acknowledged. From the work presented here (Figure 4 in particular) it appears that a parabola consists of two halves, each the degenerate case of part of a sibling curve. We are left with the question as to how this interpretation fits in with the traditional locus definition of a parabola.

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Mathematics and programming: tentative findings from a design research project

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Background: the case for programming

Mathematics is a ubiquitous and vital substrate on which our culture and societies are built. Yet this fact is seldom fully exploited in educational contexts. The first step must, in our view, be to open the black box of invisible mathematics to more people. It will become evident that the challenge is broader than it seems at first sight, as we will need to confront the necessity that mathematics itself – not just its teaching – may have to be reviewed. The experience of girls and women in relation to this challenge is pivotal. We have argued elsewhere that the complexity of teaching and learning mathematics, and will conjecture that mathematicians and mathematics education researchers can exploit digital technology to reveal more of what mathematics *is,* by offering a glimpse of the mathematical models underlying many of the systems that form part of daily life (Hoyles, Noss, Bakker and Kent, 2010).

We will illustrate the conjecture by presenting some theoretical and practical issues in design research into the role of programming in mathematical learning, based on our ongoing experience of a large-scale design research study in England, the ScratchMaths project. Mathematics and programming in schools have a longstanding and intertwined history. More generally, learning to program has been shown to be an engaging activity for most children: they become more autonomous as they build, learn from feedback and debug. Programming at the school level has been shown to have the potential to develop higher levels of mathematical thinking, in particular linked to multiplicative reasoning, mathematical abstraction, including algebraic thinking, as well as problem solving abilities (see e.g. Clements, 2000). We note however that unless activities are carefully designed, managed and sequenced, there might *not* be positive learning outcomes or trajectories, and there is an added risk that mainly advantaged learners – often boys – show learning gains (see for example, Yelland & Rubin, 2002).

New programming languages have been developed since much of the research in computing and its relation to mathematical learning was undertaken. For example, Scratch is the latest language in a fifty-year history of LISP-based environments, the most famous of which (and the most researched) is Logo. These technical developments in what it means to program and the functionalities now available offer an unprecedented opportunity to investigate how best to develop the benefits of programming for what is now known as 'computational thinking' (CT) (Wing, 2008). In addition, and the core of the research reported here,

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is to revisit the potential and challenges of exploiting programming for developing mathematical thinking and reasoning, to design an intervention arising from this analysis and to evaluate its effects on these processes and on mathematical attainment, again something now possible in England as computing is compulsory for all school students.

Constructing a relationship between programming and mathematics

In the previous section, we focused briefly on the historical research effort to understand what can be learned by programming. Our goal in this paper is much more focused and, we claim, somewhat unusual: it is to build a curriculum for learning mathematics *through* programming. By the time students come to more advanced mathematics, we traditionally assume that they are fluent in the language of mathematical expression – algebra (of course there are other 'fluencies' that matter like geometrical intuition). But we know that large numbers of students arrive at this point *without* being fluent in algebra – far from it. Our aim is to take steps towards building a mathematics curriculum of the future, one in which we may take for granted another powerful representational infrastructure to complement algebra – namely, programming. This means that we have two tasks: 1) to develop a curriculum for fostering computational thinking, introducing quite young students to the ideas of the subject and 2) build a curriculum for mathematics, in which the key ideas of the computational thinking course are exploited.

The case for programming and mathematics is nicely put by diSessa (2001) who argued that it "turns analysis into experience and allows a connection between analytic forms and their experiential implications that algebra and even calculus can't touch" (*ibid.*, p. 34). The key effort in the work reported here is to elaborate the relationship between some clearly identified core ideas of programming and their expressive power in the articulation of mathematical ideas: the research on which this is based is a three-year project, the ScratchMaths project, which we now outline.

Overview of the ScratchMaths project

The overarching aim of ScratchMaths is to develop and evaluate a curriculum for computational thinking (CT) which can serve as a substrate on which to construct a new mathematics curriculum that exploits CT, and to use this research to tease out broader issues around programming and mathematics. Our specific aim is to assess the extent to which students at Key Stage 2 (age 11) can boost their national test scores at age 11 years by participating in a two-year specially designed ScrathMaths curriculum. The first year of the project has been devoted to iterative design and development of this curriculum; the second and third to its implementation in schools. More than 100 English primary schools were recruited during the early part of 2014-15. During 2015, a huge effort was made in designing materials, followed by their iterative testing, and a focused effort of professional development for those teachers who would be involved in the trials. Details will be given at the conference.

Space constraints do not allow us to do more than flag our thinking on this ambitious aim to exploit of computational thinking concepts for mathematics. At the conference we will give some examples of how CT concepts such as algorithms, decomposition, iteration, variable and parallel processing can be designed to used to exploit reasoning about mathematical ideas such as the arithmetic of integers, variable and expressions, place value, position, ratio and other topics of the late primary and early secondary mathematics curriculum (this mapping is *not* 1-1!).

Two kinds of evaluation will be undertaken. The first will consist of a qualitative analysis of how the CT curriculum offers an expressive medium for working on the Scratch-based curriculum of the first-year implementation (the second project year) and the mathematics curriculum of the second year implementation year (the third project year). The second will be undertaken by an independent evaluator who has split the sample of 100+ schools into *treatment* and *control* groups with similar profiles grouped around 7 hubs across England, and who will use item-based evaluations of national tests in mathematics to assess the effect of participation in the curriculum on the learning of mathematics.

Design research in the ScratchMaths project

As with all design research methodologies, our work has evolved cycles of design and process evaluation. Our objective is to chart as closely as possible the triangle of interaction between the ScratchMaths curriculum instantiated in sets of activities, the teacher's interventions, and the pupils' actions and productions. Apart from allowing us to gain a picture over two years that will describe and analyse the role of the new curriculum in developing mathematical ideas, it will allow us to identify a causal mechanism for any quantitative findings that emerge from the independent evaluation. Key foci for our qualitative analysis will be i. how the 'big ideas' of ScratchMaths are communicated through the activities, ii. the resilience of the intervention to 'legitimate' and 'lethal' mutations in pedagogy and curriculum 'delivery' (Hung *et al.* (2010) and iii. the extent to which teacher expertise and experience impact outcomes.

To achieve this, we have selected 5 trial schools in London with which we work closely to:

- Observe the same key activity from each of the 3 modules (total 3 lesson observations for each Year 5 class)
- Interview 6 pupils per school (at different attainment levels) after the activity to determine what they learned/found difficult
- Interview teacher after lesson to establish their preparation, aims and assessment of the lesson as well teaching challenges
- Explore differences between the two Y5 classes in the same school and also between schools in terms of teaching approaches and adaptations of the materials.

Developing a pedagogical framework (5Es)

We have developed a pedagogical framework that we use to guide and structure the development of the materials. We call these the 5E's, and they consist of:

Explore: Investigating ideas, trying things out for yourself and debugging.

Explain: A crucial aspect of understanding ideas is being able to explain what you have done and articulating the reasons behind your approach.

Envisage: Have a goal in mind and to predict what the outcome might be *before you try it out.*

Exchange: Collaborating and sharing is a powerful way to learn- trying to see a problem from another's perspective as well as defend your own approach and compare it with others.

bridgE: Making links between the Scratch programming work and the language of 'official' mathematics.

A concluding remark

In this presentation, we will report on the framework of curriculum design (which embeds programming) that seeks to enhance both computational thinking *and* mathematical attainment, as provisional outcomes of the design research. We would also raise some key issues for research, not least the problem of how universities should revisit what they teach and how in the light of new generations of students schooled in programming and fluent in exploiting the expressive power of programming to explore mathematics.

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A reading course on Galois Theory

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I give a short report on a "best practice example", hence reflecting ideas of a mathematician towards didactics of mathematics. I will speak on my experiences of giving a reading course on Galois theory replacing the usual lecture style on this subject. This course was aimed to math majors in their second or third year at a German university. I gave the course twice with different outcome. The results may have consequences (at least for me) for the mathematics education at the beginning level at universities.

This is not a talk on a scientific paper. I will report on the realization of a reading course, and ask some question which arise in this context.

The course Galois Theory is offered for students of mathematics in the second or third year. They had previously attended a lecture „Introduction to Algebra", where they have learnt the basics on groups, rings and fields. These participants are also familiar with Linear Algebra and Analysis.

Galois Theory is the second part of the well-known lecture „Algebra" whose content hasn't changed since the days of Emmy Noether, c.f. the book „Modern Algebra" by B.L. van der Waerden. Due to the new undergraduate programs this traditional lecture is now divided into two parts. Highlights of this second part are the algebraic description of the construction with ruler and compass, as well as investigations on the solvability of algebraic equations by radicals. The tool in both cases is the correspondence between groups and field extensions.

Traditionally this course is also offered as a lecture. The lecturer presents the contents completely on the board or with a projector. In Kassel usually not more than 20 students participate. (Unfortunately teacher students avoid this subject, as well as other topics in Pure Mathematics.) Now this second part, Galois Theory, should not be offered as a lecture but as a reading course.

For this purpose, I selected the textbook „A Field Guide to Algebra" by Antoine Chambert-Loir (Springer). The reason for choosing this book was its compact presentation of the material, which could lead to the desired results in the given time. In addition the tools on groups, rings and fields are integrated in this book, which could help repeating these prerequisites. The fact that the book is written in English was regarded as an additional advantage.

During the semester in each of the 12 weeks we had a course meeting of about two hours. In addition there was a tutorium where some exercises were discussed, this was lead by an assistant. I will report only on the course meeting.

For each of the 12 meetings the students had to read and understand between 6 to 8 pages of the given book.

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During the meeting we wanted to repeat and summarize the corresponding parts of the book together in a discussion with the whole group. But before that I would of course answer questions of the students, which came up at their reading. For the discussion itself each student had to prepare two fundamental questions about the text. He or she should know their answers. With these questions I wanted to structure the discussion of the group, and I only wanted to be its moderator.

Up to now this reading course took place twice, in the summer 2013 and 2014.

Experiences in summer 2013

At the beginning of the event, the first 2 to 3 appointments, about 15 to 20 people participated. But after that only 8 students were regular participants, all of them were very active during the course.

The students were very engaged. They were always well prepared; they had read the text carefully. In particular their prepared questions provided an excellent framework for the discussion of the contents. These were real fundamental questions (e.g, „what is the definition of separable?“, “what is a normal subgroup?“, etc.).

I knew most of participants before; they belonged to the top group of their semester. Some of them were already working as tutors.

As the organizer of this reading course I had to fight with the problem of „completeness“. As a lecturer I am used to present the material completely without gaps in a logically correct sequence. All together I want to provide a complete view of the topic. Now as a moderator I had to accept that the participants made their own picture of the material and developed their own ideas of the relations. I gave this influence out of hand. Thus I had to force myself now and then not to fall back in lecturing – even if this „completeness“ seems not to be given. („Completeness“ is certainly also a problem in traditional lectures: Expectation of the lecturer – reception and acceptance of the audience.)

At the last meeting we had a discussion about this form of a reading course. I summarize this:

- The concept of this kind of reading course was valued as good by all the participants. It was mentioned positively that one could choose ones own speed in reading and could learn the material reasonably fast. However this only works with great interest and active participation. It was seen critically by a few students that the material was treated several times by reading and by discussing at the meetings; this could be boring at the end.
- All the participants could imagine this form of a course to other events. But they explicitly excluded the beginner courses, which should still take place as traditional lectures.
- It was discussed in great detail whether the order „first reading by yourself, then summarizing the content in a group discussion“ should be changed to „first a summary by the lecturer, then reading by the participants". This question was then transferred to the traditional lectures. Some students wanted here also preliminary infor-

mation (e.g. texts) before attending a lecture. They argued that in this case they could follow the lecture in an easier way. Their key point here was „preliminary working" rather than „repeating the lecture". But at this point there was no general opinion in the group.

- Much of the discussion was about the choice of the right book. The language was not a problem at all. However it was criticized that the book was too one-dimensional and presented more details than concepts. It was interesting that the books they mentioned as alternatives are very classically written in the „definition-theorem-style" with a rather encyclopaedic point of view.

Experiences in summer 2014

At the beginning of the event, the first 2 to 3 appointments, about 10 to 12 people participated. But after that only 6 students were regular participants. Four of them were actively involved, the other two were very silent and could not be really reached.

With my experience of last year, I acted at the beginning as a moderator. I assumed that the students had read the pages in the book independently. But then it was not clear to me if they had actually read the book and prepared the topic. I often doubted it, but on the other hand I was some times surprised by their detailed observations and concrete questions. They prepared the two fundamental questions seldom, and then these were not very helpful for the discussion of the material.

The students were totally confused when I summarized the content of the pages in a slightly different style, not using exactly the words of the book. That was not a problem at all one year ago.

Then I changed my attitude from a moderator to a teacher. I asked specific questions about the text, I asked for definitions and theorems, which they should have read. But these questions could not be answered. Then I made the students search for the corresponding passages in the book, read them and translate them into German. Their sentences made no sense. And it was not a question of the language; they had problems in understand the meaning of a statement. It was not clear to them what the assumptions and what the conclusions of a mathematical theorem are.

As a consequence the reading course was transformed back into a more or less traditionally lecture. And finally this also changed the attitude of the students; they expected a „complete" explanation of the topic.

At the end the final discussion about the form of the reading course was not very fruitful. The students could not reflect their own working and learning behaviour.

Conclusions and Questions

We see two completely different realizations of the reading course. The students of the first group had probably learned the reading and understanding of mathematical texts by their own. The students of the second group could not read texts and work independently, even after more than two years of studying mathematics. By the way, their grades at exams were not so bad. They were dependent on a strict guidance by the lecturer.

This raises some questions:

- Where does one learn to read and understand mathematical texts?
- Do the more and more polished beginner's lectures (with tutorials, scripts, learning centres, etc.) prevent rather than encourage the self-learning process?
- How can one influence the self-working process of the students without again guiding them too much?

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Constructionist computer programming for the teaching and learning of mathematical ideas at university level

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In the 1980s and 90s, the idea of engaging in computer programming for learning and exploring mathematical concepts became popular in mathematics education at basic levels, in what is now known as the constructionism paradigm. However, the teaching and learning of mathematics in higher levels was rarely influenced by that approach. Here I present some of the projects that I been involved with in the past few years, where this idea of students' engaging in active computer programming, within structured learning environments, has been at the centre. One such project, involves engineering students engaging in the construction of video games, where they have to use mathematical models to create simulations of physical and engineering systems. Another is a virtual mathematics laboratory for continuing education distance students where they collaborate in building and exploring models of real phenomena. A third one is an approach for the teaching and learning of statistics where both under- and post-graduate students carry out sequences of tasks with the R programming environment for them to actively engage with both the statistical concepts as well as with a tool (R) for carrying out statistical analyses.

The constructionist paradigm

When Seymour Papert and his colleagues developed the Logo programming language in the 1960s, the idea of learning by communicating (expressing) instructions to a computer was developed, and later expounded in Papert's (1980) book *Mindstorms*. This became the Logo philosophy, later termed constructionism that is described as sharing "constructivism's connotation of learning as 'building knowledge structures' [...] then [adding] the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity" (Papert & Harel, 1991, p. 1).

In Papert's (1980) vision, one particularly valuable means of achieving the above is in programming the computer because, in doing that, the student "establishes an intimate contact with some of the deepest ideas from science, from mathematics, and from the art of intellectual model building" (p. 5); and "in teaching the computer how to think, [students] embark on an exploration about how they themselves think" (p.19). However, Papert not only places emphasis on computer programming, but also on the entire learning culture where educators (often called "facilitators") can help by creating the conditions for construction and invention (rather than providing ready-made knowledge), giving students objects-to-think-with including "emotionally supportive working conditions [that] encourage them to keep going despite mathematical reticence" (p. 197).

Since the 1980s, there have been many projects attempting to implement the constructionist paradigm in order to enhance the learning of mathematics, many of the first ones using Logo programming, and most at primary or middle-school levels. However, implementing

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constructionist exploratory learning environments in school cultures is problematic and complex, as has been discussed elsewhere (e.g. see Laurillard, 2002; Ruthven, 2008). Already in 1992, following the first ICMI Study Conference (on technology) held in 1985, Burkhardt and Fraser pointed to the evidence of difficulties for teachers in implementing such learning environments, although they acknowledged that the computer provides opportunities for exploration and experimental mathematics, and that programming projects, at school and university, had shown possibilities, although programming-based activities need to be designed carefully.

The difficulties in implementation led to a significant decrease, at the end of the 20th century and beginning of this one, of both open extended work in school mathematics, as well as of computer programming (replaced by, less user-expressive, software tools with new types of interfaces). In recent years, however, there has been a renewed recognition of the importance for students to learn how to program and develop computational thinking (an example of this is the so-called Hour of Code – see code.org). And a fairly recent study by Rich, Bly and Leatham (2014) confirmed Papert's claims from 1980, by showing that programming and solving programming problems, can: provide a context for many abstract concepts; illustrate the distinction between understanding the application of mathematics in a specific situation, and the execution of a procedure; help divide complex problems into more manageable tasks; provide motivation and eliminate apprehension; and give context, application, structure and motivation for the study of mathematics. In fact, programming can be an engaging problem-solving activity where students can explore mathematics in different representations and generate and articulate mathematical relationships. Nevertheless, as has been learned, these programming activities need to be carefully designed and structured within a learning environment.

Thus, expanding on Papert's initial vision to include the lessons from the past 40 years, I consider that a constructionist implementation needs to have the following characteristics: (i) There has to be a (technological / computational) medium for an *expressive* activity (e.g. computer programming, or building/describing models or structures in a software), where the computer acts as mediating agent (computer-based activities are not meant to teach about the software, but as means for exploring and expressing ideas). (ii) Students need to be actively involved and at the centre of their constructions and explorations. (iii) The activities should take place within a structured learning environment (e.g. a *microworld,* in the extended sense described by Hoyles & Noss, 1987) that consider the characteristics of the specific learners and includes: a careful pedagogical design with materials (e.g. worksheets), and appropriate teacher interventions; and a social environment where students can collaborate and where products can be shared and discussed in small and whole groups. My perspective is in accordance with the three categories given by Resnick (1996) of *discussing constructions*, *sharing constructions* and *collaborating on constructions*; that is where students work together on design and construction activities, whether supported or not by computer networks (as promoted by Resnick's idea of distributed constructionism).

I also follow Laurillard (2002), who advocates for constructionist and collaborative technology-based learning environments in higher education, taking into account how students learn. For this, she considers that "the aim of university teaching is to make student learning possible [...] not simply impart decontextualised knowledge, but must emulate the success of everyday learning by situating knowledge in real-world activity" (p. 42) helping students reflect on their experience of the world and ways of representing it.

Mathematical constructionist implementations in higher education

As mentioned above, there are few constructionist implementations for mathematical learning at university level reported in the literature, relative to those reported at the K-12 levels. A notable exception is the MICA (Mathematics Integrated with Computers and Applications) program at Brock University in Canada (see Marshall, Buteau & Muller, 2014; see also Buteau's, 2016, contribution in these proceedings). That program was co-designed by Eric Muller, who was one of the first ICMI study participants, and it stands out as a complete curricular implementation that has been functioning for over a decade – rather than being just a limited-scope project – and that integrates computer programming activities in the pure and applied mathematics syllabi.

On my part, in recent years, I have been involved in three constructionist projects in higher education, which I present next. All three consist of carefully designed activity sequences where university students engage in computer programming and/or expressive activities for mathematical exploration or learning, that include sharing, collaboration and discussion; furthermore, in all three projects students are involved with topics and data related to real-life phenomena and that can be meaningful for their area of study.

Videogame construction by engineering students

In this project (see Pretelín-Ricárdez & Sacristán, 2015), we have been working with university engineering students, in their last year of studies: inspired by the constructionist philosophy and in particular by the work of Kafai (1995), we ask them to create videogames in which they have to use and adapt mathematical models. Some of the videogame topics involve: physical phenomena (e.g. water behaviour that needs to be modelled and simulated); navigating mazes by virtual robots (which requires using and designing digital systems, i.e. combinational logic circuits); or include simulated mechanical systems (e.g. robotic arms). Thus, students develop know-how, for their future profession as engineers, of how to apply mathematical knowledge and modelling.

The videogame constructions are structured through sequences of model-building tasks whose design takes into consideration the six principles of Model-Eliciting Activities – MEAs – described by Lesh *et al.* (2000): reality, model construction, model documentation, self evaluation, model generalisation, and simple prototype. MEAs share some of the conceptions of constructionism in that:

> *the products that students produce [...] involve sharable, manipulatable, modifiable, and reusable conceptual tools (e.g., models) for constructing, describing, explaining, manipulating, predicting, or controlling mathematically significant systems.* *(Lesh and Doerr, 2003, p. 3)*

Each activity sequence involves several stages combining or alternating paper-and-pencil work; individual and/or collaborative programming work; and whole class discussions.

The programming of videogames is a motivating activity, that engages students in producing working models of certain real-life behaviours but in a context that is meaningful to them. It also helps them gain a deeper understanding of all the elements involved in the modelling process. For example, in the cases that include water-behaviour (Pretelín-Ricárdez & Sacristán, 2015), students first need to produce a mathematical model for that behaviour: they usually come up with complex models of fluid mechanics, addressing the water model either as a molecular model or as a continuous model. They then realise that these models cannot be programmed as such into the videogame engines, so they are forced to analyse and discern the most important elements present, in order to produce, and program, simplified models into the videogame engine. They do this through collaboration and discussion.

Figure 1 gives another example of a videogame, where the student-creator recorded a table of the characteristics of the objects in the game that would need to be programmed into the game engine (in this case GameMaker Studio, http://www.yoyogames.com).

	Player	Gift Box	Motorcycle	Jeep	Gem
Density	0.2	0.01	0.5	0.7	0.8
Restitution	0.8	0.01	0.1	0.1	0.01
Linear damping	0.1	0.3	0.4	0.5	0.9
Angular damping	0.01	0.1	0.5	0.7	0.6
Friction	0.2	0.01	0.5	0.7	0.9

```
var fixture = physics_fixture_create();
physics_fixture_set_circle_shape(fixture, 16);
physics_fixture_set_density(fixture, 0.2);
physics_fixture_set_friction(fixture, 0.2);
physics_fixture_set_restitution(fixture, 0.8);
physics_fixture_set_linear_damping(fixture, 0.1);
physics_fixture_set_angular_damping(fixture, 0.01);
physics_fixture_set_collision_group(fixture,1);
physics_fixture_bind(fixture, inst_D4F006DC);
physics_fixture_delete(fixture);
```

Figure 1: A students' videogame with his table of physical characteristics of the objects involved that he would need to program into the game engine (right-bottom).

As many students have explained, the videogame construction activities also provide them with the opportunity to apply their theoretical knowledge in real-life projects and experiencing how such real-life projects could be carried out. In this way, students gain insights and expertise on how to apply their knowledge in realistic projects in different contexts related to their engineering profession.

A virtual mathematics laboratory

The second project (see Olivera, Sacristán & Pretelín-Ricárdez, 2013) involves a distance-learning environment (a virtual laboratory) where university students (mainly adults pursuing continuing education) are encouraged to explore, collaborate, build models (using various kinds of expressive software described below), discuss and reflect upon various types of real-life mathematical problems (e.g. related to linear motion; gravity and free-fall; population growth; cryptography). As in the first project described above, this project is also inspired on Lesh's *et al.* (2000) model-eliciting activities, where many of the tasks centre on building models, cycling through models and sharing these.

The investigations use different materials (e.g. real-life videos) and a variety of complementary software – e.g. video software for frame-by-frame analysis; a virtual ruler for measurements (e.g. JR Screen Ruler, http://www.spadixbd.com/freetools/jruler.htm); spreadsheets or CurveExpert (http://www.curveexpert.net) for finding mathematical equations to fit the data; and modelling software, such as Modellus (http://modellus.co), for building mathematical models and comparing them to the real data.

Interesting activities have involved the analysis of videos (which have led, for example, to extensive collaborative discussions on determining the scale of the videos), developing models to reproduce the behaviours and phenomena shown on the video, and analysing and discussing which proposed models best fit the real data (see example below).

Since students are at a distance, they need to collaborate and share their conjectures and findings online in a web-based discussion forum. We consider this useful for learning, since it forces students to express their ideas, constructions and conjectures as clearly as possible to others in written form, thus helping them clarify their own understandings (while at the same time acting as windows, for teachers and researchers, into their meaning-making – Noss & Hoyles, 1996). In most activities, we have found students collectively brainstorming on a problem or on part of a problem. They are also encouraged to propose new problems to the online community. For example, after analysing free-fall videos of dropping objects on Earth, some students proposed analysing the gravity on the Moon by analysing a NASA video of an astronaut jumping on the Moon (Figure 2).

Figure 2: Students propose, in the online forum, analysing the gravity on the Moon and creating models of an astronaut jumping on the Moon.

Figure 3: Comparison in Modellus of the real data (left-hand side) with the model constructed by the students and represented by the green dot on the right-hand side.

Because the video is not of a free-falling object, it generated discussions on what kind of movement it is (with students concluding it is a type of parabolic shoot with a nearly vertical angle). They then proposed mathematical equations that they then implemented in Modellus and compared it to their real data (Figure 3).

As in the first project described above, the construction of models and simulations helps students identify and discern the important mathematical elements in the situation under study and that help model it. Furthermore, as in the example above, the construction of meanings is also helped by the support of the social structure created by the online community.

R-based tasks for the learning of statistics by environmental sciences students

In another project (see Mascaró, Sacristán & Rufino, 2014), we have designed probability, statistics and experimental analysis courses for college and graduate environmental sciences and biology students – who tend to have strong aversions to mathematics and statistics – through sequences of constructionist and collaborative, computer-programming activities in R (see the R Project for Statistical Computing – www.r-project.org). These tasks have been directly inspired by Logo programming microworlds.

The aim is for students to develop statistical reasoning, rather than applying blindly statistical tests; build statistical models for research; apply and understand statistical computing software (in this case, the R programming language) to carry out calculations in experiments; and learn how to interpret the results given by the software. All the activities are presented through R-code "worksheets" with instructions, guidelines, examples (using data adapted from real research situations), programming tasks, questions for reflection and comments (see Figure 4).

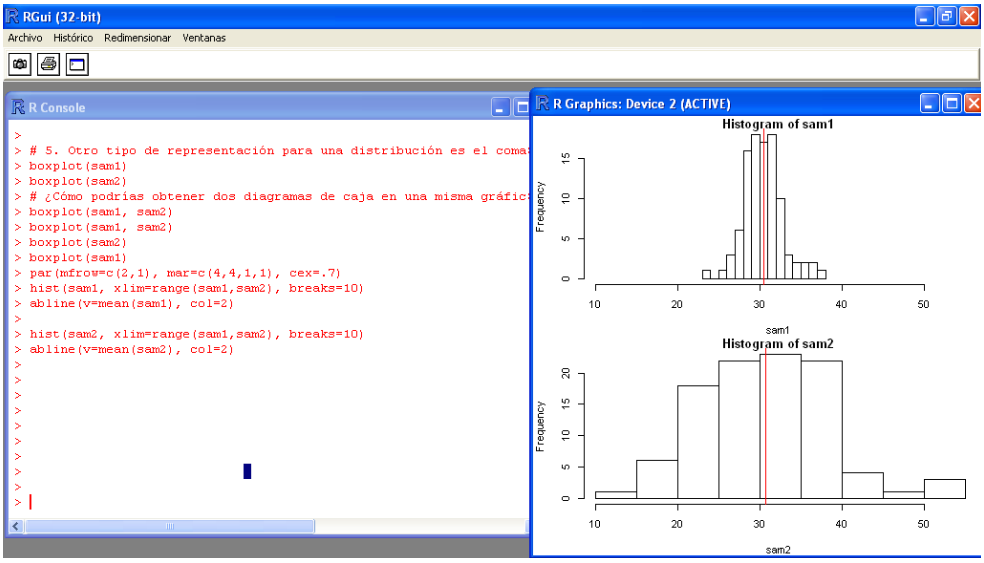

Figure 4: Part of an R-based task, including on the left-hand side some worksheet questions, with typed commands for generating graphs (i.e. the histograms presented at the right).

The understanding of statistical models is facilitated by creating objects in R, to represent them (e.g. graphs, lists of data, statistical values, etc.). Tasks are carried out through collaborative work leading to reflective interactions, explanations and evaluations. By typing R-commands, students draw and interpret graphs (i.e. visualise models) relating numerical data to graphical representations, as well as to mathematical formulae. They need to predict what a change in the programming code would produce. In this way, students go back and forth in the analysis of the data, and suggest changes for obtaining different representations.

After four years of design research and over a dozen courses at university and post-graduate levels where the tasks have been implemented and refined, preliminary results have been encouraging, particularly in the affective dimension (see Mascaró, Sacristán & Rufino, 2015): many students lose their fear of statistics, with most of them actively engaging in the activities; furthermore, several students have appropriated themselves of the software (e.g. building their own R scripts) for their own research with an apparent clearer understanding of statistical concepts.

Final remarks

All three projects described above meet the characteristics for a constructionist implementation, outlined at the beginning of this paper: at the core of each project are tasks that give students a central active role for exploration and construction, where they have to engage in some type of expressive activity (programming and/or modelling) using technology; and they all involve collaborative work and group discussions, where products are shared and analysed. We consider the latter social aspects to be fundamental for reflecting on the knowledge put into practice, and generating more stable meanings for that knowledge.

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Misunderstanding: a straight path to misconception?

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This is a report about experiences with mathematical language in student-teacher interactions especially with respect to examinations.

Language in mathematical teaching

Dealing with the particularities of mathematical language is one of the challenges for first year students in mathematics – every teacher can confirm this from experience. In the German "Bildungsstandards für die Kompetenzbereiche im Fach Mathematik" [1] six fields of competences in mathematics are identified, each in three levels of special requirements for the particular skills. Measuring the results of examinations against these standards shows that in particular in the field of competence „Arguing in mathematics" the future mathematics teachers quite often do not comply even with the lowest level. In addition, one of the more depressing observations in this context is that for quite many students even five years at the university do not change this.

In order to get a better grip on the particular difficulties of the students I started to collect answers from written and oral examinations. The first aim was to analyze the typical misunderstandings. The next aim is of course how to deal with them and how to avoid them. The data were collected in two fields:

- Explanations and definitions for the main terms of a first year Analysis course
- Arguing in elementary number theory

What does it mean: recall a definition?

In my written examinations of the first year analysis I often observed that students quite often are not able to apply definitions even to simple examples. Therefore I started to include one task related to the reproduction of the central definitions and the relations between them.
To this end various formulations were used, here examples related to Analysis I and II, respectively:

- Convergence
    - Explain ("erkläre") / explicate ("erläutere") the concepts ("Begriffe") Cauchy sequence and convergent sequence and their relationship.
    - What is a convergent sequence? What is a Cauchy sequence? What is the difference/relation?
    - Complete the following definitions: A real sequence $(a)_n$ is convergent....
- Derivatives
    - Exlain/explicate the terms derivative, directional derivative, partial derivative. How are they related?

- Let $f: U \to R$ be a given function. What does it mean: the function is differentiable/partial differentiable/ possesses directional derivatives in $a \epsilon U$?
- Complete the following definitions: A function … is differentiable in $a \epsilon U$ if and only if ….

The answers were highly dependent on the specific formulation of the questions. Apart from the usual mistakes in recalling definitions like wrong or interchanged quantifiers, e.g., in both examples students mostly understood the first two formulations as a request to give some explications of their own mind rather than the correct mathematical definitions. In some cases it happened that they reproduced the correct definition and then added a wrong explanation indicating one of the common misconceptions. The latter happens very often while asking for the definition of a convergent sequence. Another phenomenon is a certain insensitivity about what is needed for a proper definition. Typical answers in this direction: A sequence is convergent if $lim_{n\to\infty} a_n = a$ or a function is partial differentiable iff all partial derivatives exist – without giving the definition of a partial derivative.

The next example is taken from a series of oral examinations (teacher students for secondary schools) here the students knew in advance about the question: What is a convergent sequence? More than 50% answered: A sequence is convergent if it tends ("strives to reach" "strebt") to a limit value but never reaches it. Especially the second point was very important to them. When asked about the references for this information the sources internet, math-lessons from high-school and math-textbooks for high-schools were mentioned in equal parts. Without any doubt you can find these "definition" on internet platforms but the students mentioning textbooks were also absolutely convinced about their source.

Comparing and distinguishing between definitions

When going through the central definitions of a first year Analysis course questions related to integration are less popular than questions related to derivatives, e.g. At the end of the Analysis I course usually more than 50% of the students have no proper answer to the question what an integrable function is. This quotient is even higher when the Cauchy-Riemann integral ("Regelintegral", approximation via uniform convergence of step functions) is introduced in the lessons instead of the Riemann integral. In addition also high emphasize on the difference between the notation primitive and integral and the role of the main theorem of calculus does not prevent that these terms are considered synonyms more or less.

Again the answers depend on the particular formulation of the question. During my last Analysis courses I always introduced the Riemann integral via the upper and lower Riemann-Darboux integral. At the end of my last Analysis course the formulation of the corresponding task was:
Let $f: [a, b] \to \mathbb{R}$ be a bounded function. What does it mean: f is Riemann integrable over $[a, b]$? Give two sufficient criteria for Riemann integrability.

Correct answer: The function is R-integrable it the upper and lower integral coincide. (Of course highly appreciated: an explanation for these terms)
Expected criteria: continuity and monotonicity or the Riemann criterion, even in the form:

the difference between upper and lower sums becomes small if the partition of the interval is sufficiently fine.

Qualitatively the resulting failures or inaccuracies in the answers were similar among all groups of students, here two groups of teacher students TSL3 (high school teachers), TSL4, mathematics bachelor MB, physics bachelor PB) although the quantitative view gives some differences as the following tables tell:

Definition	TSL3	TSL4	MB	PB
None at all	17	12	6	4
The upper and lower <u>sum</u> coincide/coincide for some ε confusion between upper integral and upper sum	1 2	1	1	1
The function must have a primitive	4	3	5	
other	8	1		

Table 1: Definition of Riemann integrability

Criteria	TSL3	TSL4	MB	PB
None at all	12	1	3	4
Continuity is sufficient	13	6	8	4
Monotonicity is sufficient		1	2	1
Riemann criterion	2		2	1
Differentiability is sufficient	7	5	2	1
Continuity is nessecary	6	3		1
Differentiability is necessary	6	8		1

Table 2: Criteria for Riemann integrability

It is remarkable that 59% of the teacher students did not even try to answer the question for the definition of the Riemann integral. Concerning the criteria obviously monotonicity is not a criterion of which the students are really aware. In contrast the majority associates (confuses?) differentiability with integrability. What was not (and could not) be tested in this context whether the correct argumentation was clear for the students, namely that differentiability leads to continuity hence to integrability.

What is also obvious that a significant part of these students has a poor feeling for language at least about distinguishing between necessary and sufficient conditions. The formulation: "Die Funktion muss differenzierbar sein – the function must be differentiable" could be a slip of the tongue or an indicator for a missing comprehension of the relations.

Arguing in elementary number theory

Another class of examples is taken from examinations of L2-teacher students (secondary school). Here usually the students are less educated and less motivated for mathematics and have more difficulties with abstract concepts than L3-teacher students. Nevertheless they should also comply with the education standards at least at a low level even from the start of their university studies. Here the test problems partly consist of explaining definitions and relations in elementary number theory and again the particular formulations lead to obvious misunderstandings. Here one example, three questions in a row:

a) What does the theorem about division with remainder tell?
b) Which fractions can by expressed as a finite decimal number and why?
c) When and why is a natural number divisible by 4?

Expected answer to c) was the divisibility criterion related to the last two figures of the number together with the justification (multiples of 100 are divisible by 4). However, quite a large number of students did not understand the question in this way, typical answers were: if it is a multiple of 4 or: if there is no remainder left after dividing through 4. Of course then typically there was no answer to the question "why". In the next problem of this test negations and elementary justifications in the context of divisibility were required. It becomes even more apparent from the answers to this task that a significant part of these students is quite averse to precise mathematical formulations – a language which they experience as strange and not adequate for their understanding. Hence it is not clear whether the formulation of c) in the form: "Give the criterion for divisibility of a natural number by 4 and a justification" would have encouraged more correct answers.

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3. MATHEMATICS AS A SERVICE SUBJECT (IN ENGINEERING AND ECONOMICS)

Differences between the usage of mathematical concepts in engineering statics and engineering mathematics education

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In this contribution we present the results of an investigation of how mathematical concepts are introduced and used in engineering statics in comparison with engineering mathematics. For this, two widespread textbooks in engineering statics have been analysed and some remarkable differences have been found regarding the usage of vectors, 'differential' elements, and variation concepts and notation. These differences create potential cognitive barriers for students which prevent them from connecting insights from the mathematical education and from engineering statics. We suggest some educational measures to address the problem.

Introduction

In engineering study courses, students encounter mathematical concepts and procedures not only in their proper mathematics education but also in application subjects that run in parallel or later. For providing an integrated curriculum as advocated by the European Society for Engineering Education (SEFI) in (Alpers et al. 2013), the different usages of mathematics should be interconnected such that students experience a study course as a *sense-making network* of modules. In order to recognize potential cognitive barriers, the use of mathematics in application subjects must be investigated and compared with the treatment provided in mathematics education. For the advanced subject "signal analysis" this has been done by Hochmuth et al. (2014) and interesting differences have been found. In this contribution we investigate engineering statics which is a fundamental subject occurring in many engineering study courses. Two widespread textbooks in engineering statics from different 'educational cultures' (from Germany: Gross et al. 2013; from the US: Hibbeler 2012) have been analysed. In the next section we present some remarkable examples for deviations from the usual treatment in mathematics textbooks (see e.g. Meyberg and Vachenauer 2001; Papula 2007). Subsequently, we make some suggestions on how these deviations could be addressed in mathematics and/or application education.

Differences in usage of mathematical concepts and notation

The document analysis of the textbooks revealed essential differences in three areas: the concept, construction and notation of vectors; the usage of differentials; and the concept and notation of virtual displacements.

In the statics textbooks, vectors are introduced as quantities which have a magnitude and a direction, not as elements of an abstract vector space given axiomatically. This, however, does not create confusion since in engineering mathematics textbooks like the two books cited above this is done similarly. In more advanced textbooks like Meyberg and Vachenauer

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(2001), only later in the exposition a more general treatment of vector spaces over the reals follows.

In mathematics, vectors are generally “free” vectors (equivalence classes of arrows which can be mapped onto each other by translation), whereas force vectors in statics are introduced as “bound” vectors which can be moved along a line of action without changing the static effect (also called “sliding vector” in Gross et al. 2013). This has consequences for vector operations since if one adds force vectors not having the same line of action as one adds free vectors, then one still has to determine the line of action of the resulting free vector, i.e. to turn it into a sliding vector again.

A larger source of potential confusion in the analysed statics books is the use of the terms “component”, “coordinate” and “absolute value” (=“magnitude”) and the corresponding notations. A vector is written as a boldface capital letter (like $\boldsymbol{A}$) and its absolute value is written as the same letter in italics (like A). If a coordinate system is given, then a vector can be decomposed into its components $\boldsymbol{A} = \boldsymbol{A}_x + \boldsymbol{A}_y + \boldsymbol{A}_z$. It is also represented using coordinates and unit vectors as $\boldsymbol{A} = A_x\boldsymbol{e}_x + A_y\boldsymbol{e}_y + A_z\boldsymbol{e}_z$. This creates an inconsistency since the symbol A_x is used to denote both the absolute value of the component vector $\boldsymbol{A}_x$ and the x-coordinate of the vector $\boldsymbol{A}$ which might be negative. This generates a need for clarification in later work on tasks for example on computing forces of pins in trusses. Gross et al. (2012, p. 42) seem to grasp the problem when stating: “We would like to point out that the quantities S_j are the forces in pins or ropes which are positive in the given direction, they are *not* the absolute values of the vectors $\boldsymbol{S}_j$” (translated by B.A.). They (and similarly Hibbeler 2012) often solve problems where forces are applied to a rigid body and unknown forces are to be determined by setting up first a unit vector $\boldsymbol{u}$ for an unknown force $\boldsymbol{F}$ with a known line of action and then writing $\boldsymbol{F} = F\boldsymbol{u}$, where F then is the unknown (signed) coordinate in direction of vector $\boldsymbol{u}$ and *not* the absolute value of vector $\boldsymbol{F}$. In essence, one has to distinguish between writing a vector as $\boldsymbol{A} = A\boldsymbol{e}$ where A is the absolute value and $\boldsymbol{e}$ is a unit vector having the same direction as A, and as $\boldsymbol{A} = A\boldsymbol{u}$ where $\boldsymbol{u}$ is a unit vector with the same or opposite direction as $\boldsymbol{A}$ and A is the coordinate in that direction.

A well-known difference between expositions in engineering and mathematics textbooks is the use of differentials (“infinitely small quantities”). This occurs in statics when defining and computing certain properties of geometric objects like first and second moments as well as centers. Here, from an object one picks a part which is infinitely small at least with respect to one dimension. Then, a (finite) property is determined (like the distance to an axis), multiplied with an infinitely small length, area, volume or mass of the part (dl, dA, dV, dm) resulting in another infinitely small property, and finally all these infinitely many infinitely small quantities are “summed up” by integration which gives the (finite) property of the whole object. In mathematics education, similar problems appear when computing properties of “curved” geometric objects like the area between the graph of a function and the independent axis or the volume of objects generated by rotating a graph. Here, an approximation with finite objects is performed and subsequently the limit is taken, and then it is pointed out that the result depends on the existence of the limit.

Differentials also appear when investigating the connection between load, shear force and bending moment regarding beams (Gross, p. 185/186; Hibbeler p. 386/387). In Gross et al. (2013) an infinitely small part is picked. Since this also must be in equilibrium one can set up the respective equations using differentials and compute differential quotients. This way it is shown that for line loads the derivative of the shear force (resp. bending moment) is the negative load function (resp. the shear function). In the book by Hibbeler, however, a *finite* object is picked, the equations of equilibrium are set up, and only later the limit is taken which is the way it would have been done in mathematics. Then, the question of how to deal with a product of differentials (i.e., neglect it) does not come up. The treatment of hanging ropes (called "cables") in the textbook by Hibbeler (2012, p. 397) is similar. In belt friction, however, Hibbeler (2012, p. 447) starts with picking a differential and sets up equilibrium equations for this. Subsequently, he uses the argument that for infinitely small arguments $sin(dx) = dx$ and $cos(dx) = 1$ which again raises the question of what is "allowed" when dealing with differentials.

The largest gap between notation and reasoning used in mathematics and in statics occurs in the area of so-called "virtual work". Here, a partial configuration is cut free such that it becomes moveable and the forces and moments acting on it from outside are identified. When these forces and torques result in so-called "virtual displacements", then "virtual work" is performed. Since the configuration is static this work must be 0. The latter is called an "axiom" in Gross et al. (2013, p. 229) which is equivalent to the equilibrium conditions such that one can assume in statics one or the other. In Hibbeler (2012, p. 555) it is stated and motivated that in equilibrium virtual work must be 0. For engineering students who usually do not know the axiomatic structure of statics, this simply means that it is allowed to use this as a fact. In Gross et al. (2013, p. 227), and similarly in Hibbeler (2012, p. 555), virtual displacements and rotations are introduced as having three properties: They are "imagined" (not existing in reality), infinitely small, and geometrically possible (i.e. the configuration is moveable). In order to distinguish these entities from infinitely small "real" displacements a special notation is used: δr or $\delta\varphi$; correspondingly, virtual work is denoted by δW which is computed by taking the product $\delta W = F^{\star}\ \delta r$ or $\delta W = M^{\star}\ \delta\varphi$. This notation comes from variation theory in mathematics which is too advanced for being included in the standard mathematics education for engineers. In Meyberg and Vachenauer (2001) there is a (final!) chapter on this topic which can hardly be assumed to be included in regular teaching. Moreover, the Gâteaux variation defined there uses a different notation δf which is already the derivative and not an infinitely small quantity.

For students to be successful in statics it is important to understand how they can use the principle of virtual work in order to solve equilibrium problems and when it is advantageous to do so instead of setting up systems of equilibrium equations. The practical meaning of having different symbols dr and δr is that they cannot be "cancelled". For example, if two deplacements are dependent, say $s = s(r)$, then the virtual displacements can be related by differentiation: $\delta s = (ds/dr)^{\star}\delta r$. If there is one possible virtual displacement δr, then one finally ends up with an equation of type $\delta W = (\dots)^{\star}\delta r$. Here, it is argued that the bracket must be zero since δr is non-zero (Hibbeler 2012, p. 556; Gross et al. 2013, p. 229). Such an argument can hardly be followed since it is unclear which properties of the reals can be used when dealing with infinitely small quantities. This becomes even more mysterious when there are two (or more) independent virtual displacements and it is argued that from the equation $\delta W = (\ldots)^* \delta r + (\ldots)^* \delta\varphi$ and the independence of δr and $\delta\varphi$ it follows that both brackets must be 0 (Gross et al. 2013, p. 235). In (Hibbeler 2012, pp. 559, 564) this problem is circumvented by recommending to consider just one displacement at a time but it remains unclear why this is allowed. In variation theory, which this is based upon, the corresponding argument is that the brackets are the Gâteaux variations which must be 0 but this theory is not available.

Potential educational consequences

The document analysis of widespread statics textbooks shows that there is potential for cognitive mismatches between students' learning in mathematics and in statics. For a mathematics educator, a first consequence of this situation should be to discuss the issues with engineering colleagues teaching statics in order to find out where such mismatches occur. The potential sources for problems that have been identified can be addressed in different ways. Issues like different types of vectors and different ways of writing a vector should be addressed explicitly in mathematics education. One could, for example, provide tasks where vectors are to be written in different ways (as absolute value times unit vector having the same direction and as factor times unit vector where the latter has the same or the opposite direction). It is questionable whether the approach in Meyberg and Vachenauer (2001) to formalize the concept of bound vectors really helps since there is the danger that the formal mathematical apparatus blurs the conceptual kernel.

The issue of using of differentials can also be explicitly addressed in mathematics education by presenting the engineering use as a kind of "shortcut": In mathematics a precise argumentation is given by first considering approximations with finite objects and then taking limits (if they exist); the engineers shorten this and implicitly assume that all occurring limits exist.

When topics are mathematically too advanced like variation theory (or distribution theory discussed in Hochmuth et al. 2014), then it does not help to provide "pseudo-arguments" that cannot be understood (like the bracket must be 0 since the differential is non-zero). It seems to be more helpful to clearly state the rules and usage scenarios. This is presumably what students concentrate on in any case because such a behavior promises success when encountering a similar type of task in examinations. A related problem occurs when advanced concepts are needed in application subjects before they are learnt in mathematics education (cf. Hennig and Mertsching 2012). If it is possible to provide a preliminary understanding tied in with existing knowledge (e.g., explaining multi-dimensional integration as infinite sum of infinitely small quantities as in one-dimensional integration) then this might be promising. Whether the measures suggested above really help in enhancing student understanding and avoiding confusion must be investigated in further research.

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Unpacking procedural knowledge in mathematics exams for first-year engineering students

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For unpacking procedural knowledge in mathematics exams for engineering students, we distinguish knowledge in mind (knowing of the web of procedures and steps) and knowledge in use (including accuracy and speed, but also flexibility and adaptivity challenged by complex situations). Although both components are required in procedural items, students' sources of failure vary with their achievement level and semester, as the empirical analysis of written tests shows. The suggested distinction and the empirical findings are relevant for adaptive training programs as well as for developing a theoretical foundation on the interplay of conceptual and procedural knowledge in tertiary mathematics for engineering.

Adapting the research program of unpacking procedural knowledge

Mathematics courses for university students in engineering have often been criticized for prioritizing procedural knowledge against conceptual knowledge, relying on the classical distinction by Hiebert & Lefèvre (1986). Thus, many research or design projects on mathematics for engineers focus on increasing the emphasis of conceptual knowledge. The high practical relevance of finding a better balance between conceptual and procedural knowledge is affirmed by studies of demands in exams, showing that 70-80% of the items refer only to procedural knowledge (Bergqvist, 2007; replicated in an own analysis of 8 exams for engineering students from four German universities, cf. Altieri, in prep.).

However, the compensatory emphasis on conceptual knowledge has led to scant *academic* attention on procedural knowledge in tertiary education, although deficits in procedural knowledge are prominently discussed in *practical* discourses among lecturers. A similar gap between the academic discourse (focused on conceptual knowledge) and the practical focus on procedural knowledge (without theoretical and empirical base) has been criticized for primary and secondary education 30 years ago, followed by very insightful research programs to unpack aspects of procedural knowledge for primary and secondary school students (Schneider, Rittle-Johnson & Star, 2011). In this paper, we plead for the need to analyze more carefully different procedural demands and their connection to conceptual knowledge in order to substantiate research and design for enhancing both, conceptual and procedural knowledge for engineers.

Hiebert and Lefevre's (1986) classical definition of procedural knowledge as "rules, algorithms or procedures used to solve mathematical tasks" (p. 6) has often been reduced to distinguishing know how and know what, with conceptual knowledge being the superior type. In contrast, Rittle-Johnson, Star and Durkin (2012) and others started research programs for secondary schools in which different components of procedural knowledge could

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be discerned, e.g. the knowledge of procedures, but also accuracy, speed, automatization, flexibility (cf. Star, 2005; Schneider et al., 2011). This unpacking allows to specify different qualities *within* the procedural knowledge rather than entangling type and quality.

Procedural knowledge-in-use and knowledge-in-mind as sources of failure

Our research contributes to adapting the research program of unpacking procedural knowledge to tertiary education, here mathematics for engineering first year students. The first step of the program starts by analyzing the different sources of engineering students' errors and their change during the first year. Usually, systematic errors are distinguished from careless errors where the systematic errors are traced back to lacks in the knowledge of procedures and underlying misconceptions (Kersten, 2015). Our exploration of engineering students' sources of failure shows that so-called careless errors can refer to all other relevant aspects of procedural knowledge: missing accuracy, speed (which is connected to missing automatization), but also flexibility and adaptivity (because using the standard procedure where shortenings are possible costs time and increases the chance for calculation errors). This is illustrated by typical student errors while applying Horner's method (in Fig. 1): Whereas Student A doesn't know how to perform the procedure, student B performs principally well but miscalculates the fractions and cannot recalculate it within the limited time.

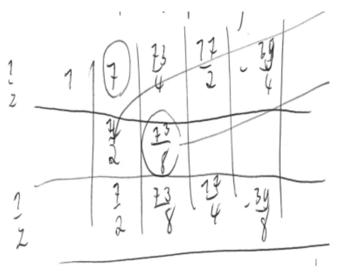
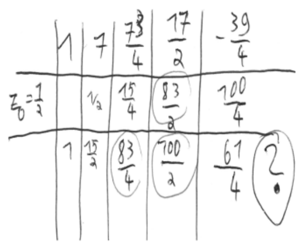
Item Horner's method	**Solution of Student A**	**Solution of Student B**
The polynomial $P(z) = z^4 + 7z^3 + \frac{73}{4}z^2 + \frac{17}{2}z - \frac{39}{4}$ has the roots $z_1 = \frac{1}{2}$, $z_2 = -\frac{3}{2}$ and $z_3, z_4 \in \mathbb{C}$. Determine z_3 and z_4 by Horner's method and calculate $z_1 - z_2 - z_3 - z_3$. [*Multiple choice between 10 results; Time limit 7:30 minutes*]		

Fig. 1: Examples for sources of failure in "knowledge in mind" (student A) and "knowledge in use" (student B)

Knowing how to perform a procedure correctly, we define as *procedural knowledge in mind.* In contrast, accuracy in performing procedures within an adequate time, we define as *procedural knowledge in use,* which comprises several demands especially in more complex situations. This component is connected to automatization and flexibility as these components can simplify the procedural demands. The conceptualization of *knowledge in use* refers to a situated theory of cognition (Brown et al. 1989), explaining why *knowledge in mind* cannot simply be applied in each situation. Whereas Brown et al. (1989) mainly refer to conceptual knowledge, a situated perspective on procedural skills is required for unpacking important sources of failure, especially failure of low performing students. The presented study unpacks procedural knowledge and compares between performance groups and longitudinally with respect to *knowledge in mind* and *knowledge in use*, guided by two research questions: (1) How does procedural knowledge develop within the first two semesters in different performance groups with respect to selected basic procedures? (2) How do these differential developments in three performance groups vary for *knowledge in mind* and *knowledge in use*?

Methods

The longitudinal study was conducted with N = 1197 students in the first year courses mathematics for engineering. In the first and second semester, the students' procedural knowledge was measured by two tests in two parallel versions each (one example of the 11 items is printed in Fig. 1). The internal consistency of Cronbach's alpha reached .74.

The test construction and data analysis operationalized *knowledge in use* and *in mind* as follows: Time restrictions were set for each item for grasping the speed component of *knowledge in use*. Two independent rater teams coded all 2×11×1197 students' written answers either as correctly solved or erroneous with respect to the sources of failure: a lack of *knowledge in use* versus *knowledge in mind* (as defined above). Interrater reliability reached an average Cohen's kappa of .66, with an expert team judging on conflicts later.

In the data analysis of each item, performances and rates of sources of failure were compared between the performance groups and in longitudinal perspective. The placement for performance groups was conducted with respect to achievement in the first test: Participants with ≤ 2 correct items were regarded as low performers, with ≥ 6 correct items as high performers and the group in between as medium performers.

Results: Different sources of failure for two selected items

In the eleven items of the first test, on average 34% of students reached a correct solution, 11% made an error traced back to *knowledge in mind*, and 55% back to *knowledge in use*. For unpacking this overall result in more detail, we present the longitudinal comparison of the three performance groups for two selected items in Figure 2. The solution rate of both items develops strongly in almost all cases. Hence, all three performance groups increase their procedural knowledge significantly within the first year of study. But low performers are not able to catch up the initial difference, their solution rates in the second test do not reach the solution rates of medium performers in the first test. In other words, the distance between low and medium performers regarding procedural knowledge counts more than one semester.

Bars 7 and 8 in Fig. 2 show that low performers first enhance their *knowledge in mind,* which leads to a better performance in procedural knowledge. But they do not necessarily equally quickly strengthen their *knowledge in use*. Bars 2 and 8 reveal that even in the second test, more than 10% of the low performers don't know how to handle Horner's method and more than 30% are not able to carry out the product rule correctly, resp. Nevertheless, bars 1 to 4, 9 and 10 show that errors traced back to *knowledge in use* can decrease substantially among medium and low performers.

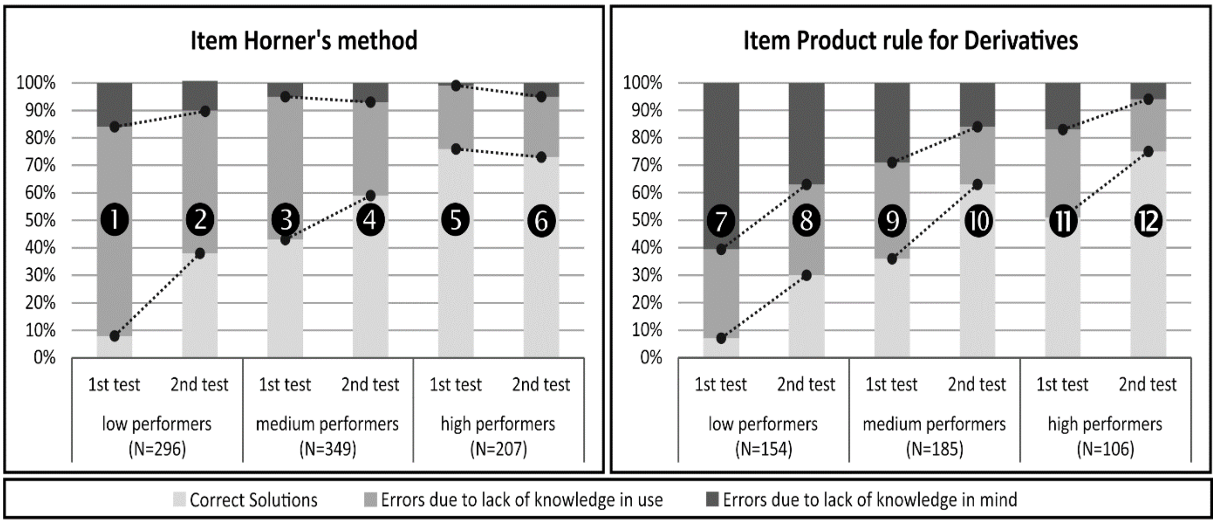


Fig. 2: Development of correct solutions, lack of "knowledge in use" and "in mind": Analysis for two exemplary items

Discussion and consequences for interventions

In this paper, we contribute to a longer-term research program on unpacking and fostering mathematical knowledge for engineering students. The distinction between procedural *knowledge in use* and *in mind* is powerful as it subsumes many important components of procedural skills (speed, accuracy, but also flexibility), and allows to isolate the component *knowledge in mind* which has the strongest connection to elements of conceptual knowledge. The mathematics engineering courses aim at both, conceptual knowledge of main concepts and connections, but also adequate procedural knowledge: Every student should know how to perform basic procedures because they are important tools in STEM fields like mechanics or signal and system theory. The study shows that *knowledge in use* and *knowledge in mind* show different dynamics for each performance group and vary significantly between items and topics.

These results are highly relevant for practical purposes because different approaches for fostering each component of procedural knowledge are required. Whereas an improvement of *knowledge in use* can be reached by strengthening automatization and flexibility, *knowledge in mind* might develop effectively by a permanent application of basic procedures in complex situations in order to routinize and flexibilize their processing with links to conceptual knowledge, metaprocedures and the web of procedures they are embedded in.

A lasting high lack of *knowledge in mind* among low performers seems to block the development of *knowledge in use* during the first year. As a consequence it should be monitored and – if necessary – refreshed frequently. Regarding *knowledge in use* our findings show that it can develop strongly in all performance groups. But at all times low performers permanently lag behind medium performers for more than half a year. This should implicate an attempt to improve starting conditions for low performers by an adaptive training program at the beginning of study.

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Motivating mathematics for biology students through modelling

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This paper describes collaboration between two centres of excellence in higher education in which mathematical modelling tasks are introduced to biology students as a means of motivating students to engage more deeply in mathematical studies. A mathematics teaching developmental pilot study is described, and attention is given to students' affective responses – their motivation to engage in the tasks and in mathematics. The responses suggest that inclusion of mathematical modelling in authentic situations may have a positive impact on these students' motivation to study mathematics.

Introduction

The first centre of excellence in higher education (CEHE) in Norway was established in Teacher Education at the beginning of 2012. Following this, three new CEHEs were created at the beginning of 2014, one in Music Performance, one in Biology and one in Mathematics. In this paper we report the efforts of two of these centres: Centre for Research, Innovation and Coordination of Mathematics Teaching (MatRIC based at the University of Agder – UiA) and Centre for Excellence in Biology Education (bioCEED based at the Universities of Bergen and Svalbard – UiB) to collaborate in developing mathematical modelling for undergraduate biology students. Collaboration across departments is not so unusual; here the collaboration between MatRIC and bioCEED is both across departments (mathematics and biology) and across institutions (UiA and UiB) separated by about 8 hours by road or three hours by air and road.

At the outset there was a desire to explore ways in which the CEHEs in biology and mathematics might collaborate. The focus, mathematical modelling, was a natural choice because MatRIC had identified modelling as an approach to motivating students' engagement by making mathematics more meaningful in their programmes of study. The biologists were also concerned to motivate students to extend their mathematical knowledge beyond the basic compulsory course in their undergraduate studies. Biology students at UiB take one compulsory mathematics course, which comes in the first semester, and students from about twenty different natural science programmes come together for a general mathematics course. There are few opportunities for focusing on the issues specific to biology.

Mathematics and biology

> *"After a century's struggle, mathematics has become the language of biology" (Steen, 2005, p. 22)*

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Recognition of the need to develop the mathematical competencies of biology students is not a Norwegian phenomenon, we note the arguments expressed, for example, in the USA. It is asserted that students of biology in the 21^{st} century need to develop mathematical competencies to meet the demands of biological science (Labov, Reid & Yamamoto, 2010). The editors of Cell Biology Education assert that "The need for basic mathematical … literacy among biologists has never been greater" (Gross, Brent & Hoy, 2004, p. 85). Further, the mathematician Lynn Arthur Steen argues that biological and mathematical sciences need to be integrated within undergraduate education (2005); Louis Gross offers an explanation about what such integration might mean: "concepts from biology should be integrated within the quantitative courses that life science students take, and quantitative concepts should be emphasized throughout the life science curriculum" (Gross, Brent & Hoy, 2004, p. 86). A report arising from a conference in 2009 organized by the American Association for the Advancement of Science draws attention to the important role of mathematics in biology and that mathematical modelling is a basic scientific skill within the 'core competencies and disciplinary practices' of biology (Brewer & Smith, 2011, p. 17).

A teaching development project

We report from the pilot phase of a developmental research project. In this research design cycles of developmental activity (planning, implementation, reflection, feedback) are theoretically informed and contribute to the development of theory (Goodchild, Fuglestad & Jaworski, 2013). This 'pilot' phase is concerned with the feasibility of a project in which teachers and researchers from one university work with students at another, and the collaboration across mathematics and biology departments. We want to explore how students receive and react to the introduction of mathematical modelling in biological situations, and generate evidence that will support the continuation of the project by convincing both mathematics teachers and biology teachers of its value.

We have met with two groups of students. The first group was a feasibility study that took place in April 2015. Ten students volunteered to participate, all but one were in their first year and had completed the first mathematics course (10 ECTS points) successfully, the exception was a second year student who had accumulated 25 ECTS points in mathematics. This group met on one occasion, for three and a half hours. The second group comprised nine volunteers. The plan for this group is to meet on four occasions during their first (autumn 2015) semester concurrently with their compulsory mathematics course. The second group is both younger and less experienced mathematically than the first group. Both groups are very small and we do not presume to draw any generalizable conclusions from these meetings.

In the presentation we hope to report from all four meetings of the second group, however at the time of writing only the first meeting has taken place. The content of the first meeting of the second group was intended to be similar to that of the meeting with the group that met in April. However, due to a two and a half hour flight delay the first meeting of the second group was shorter than intended – a significant practical issue when working across institutions!

The sessions began with a brief introduction to the basic ideas of mathematical modelling and the modelling cycle through iterations that were initiated by an authentic problem, through formulation, analysis, interpretation, validation, review of assumptions and reformulation etc. Following this, students were asked to collaborate in modelling problems of increasing complexity, but requiring only pre-calculus mathematics.

Problems set included, for example, one that required the estimation of a rabbit population based on counting the number of dead rabbits (due to road kill) along a highway. Another sought to investigate the sustainable ‘harvest’ from a population of fish in fishery.

In this report we focus on the questions:

> What is the students’ response to mathematical modeling tasks in biological contexts and their motivation to pursue studies in mathematics?
>
> What is learned about biology students’ attitudes towards mathematics?

To address these questions were refer to students’ responses to questionnaires comprised of likert style and open questions that were completed at the beginning and end of the session. We report from the likert scale questions here.

Observations and discussion

Before engaging in the activities students were asked to rate their experience of mathematics as interesting and enjoyable, to express their opinion about the importance of mathematics in biology, the relevance of their mathematics course, and the sufficiency of their mathematical knowledge. The average responses from the two groups of students, (where ‘5’ represented the strongest positive response) were as follows:

Mathematics is … interesting (3,1 and 2,89); enjoyable (2,9 and 2,56) important in biology (3,5 and 3,33), the sufficiency of their existing knowledge of mathematics (3,7 and 3,56).

We emphasise that the groups are very small and little can be concluded from these figures, but we note the relatively weak commitment and enthusiasm in their attitudes towards mathematics, slightly stronger feelings about the importance of mathematics to biology and the perceived sufficiency of their own knowledge of mathematics. The latter is of interest given that the second group is less than half way through their mathematics course. However, the question seeking their opinion about the relevance of their mathematics course to biology elicited average responses (2,1 and 3,67). Given the consistency of the earlier responses the difference here is noticeable; the students who have completed the university course are less convinced than the students currently studying the course. It is impossible to say whether this is a systematic difference and even if it were, there could be many reasons for the difference. However, the issue we want to explore further is whether the general content of the university course in mathematics influences negatively students’ opinions about the relevance.

At the end of the session students were asked whether they found the activity interesting, enjoyable and challenging; also whether the activity had contributed to their understanding of mathematics, biology and applications of mathematics to biology, and whether such activities would be useful in their regular mathematics courses. The results here show a similar pattern (5 point scale, '5' strongest positive).

The activities were interesting (4,4 and 4,13); enjoyable (4,0 and 3,88); and challenging (4,7 and 4,5). To the questions relating to increase in understanding, of mathematics (3,8 and 3,38); of biology (4,1 and 4); and applications of mathematics to biology (4,3 and 4,38). We find the strength of the positive response to the last two items surprising given that the people leading the activity are not biologists. Here also there was one question in which there was a noticeable difference in responses: to the question about the usefulness of the modelling activity in regular mathematics classes the first group responded more positively than the second (4,4 and 3,38). Again we emphasise that there may be many reasons for this difference, but taken in conjunction with our interpretation of responses to the question of relevance of their course to biology it may be that the incorporation of mathematical modelling activities might have a positive effect on students' attitude towards the subject.

We also noticed that the first group of students appeared to engage more productively and successfully with the tasks given. This could be because the first group of students had completed the whole mathematics course, some five months before the modelling session. It could also be that mathematical modelling requires students to have acquired a level of maturity as suggested by Edelstein-Keshet (2005). She suggests that mathematical modelling be included in courses for biologists as a second or third year topic "when the level of maturity of undergraduate students has increased" (p. 69). As we propose mathematical modelling be included in the first semester mathematics course we do not challenge Edelstein-Keshet's suggestion, we rather confront the challenge if mathematical modelling is introduced to biology students in their first semester.

A further objection may be raised. In our pilot experiments we are working with very small groups of students. How then can we propose that the activities be transposed to very large groups combining students from many different programmes of study? This question lies beyond the scope of the present paper, but we refer first to a meta-analysis of 39 research based studies (published in the period 1980-1996) of small-group learning by Springer, Stanne and Donovan (1999). The studies included in the meta-analysis demonstrated that learning in small groups had a positive effect on students' achievement, retention and attitudes. In another paper, Allen and Tanner (2005) point to evidence of successful transposition of problem-based learning from small group settings to large and very large classes. Our pilot project with small groups of students can be adapted to the large class settings. Our task is to convince the regular mathematics teacher that this is an effective approach.

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Links between engineering students' and their teachers' personal relationship with mathematics

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In Hernandes Gomes & González-Martín (2015), we investigated how the background (mathematics, engineering, etc.) of mathematics teachers in engineering programs can shape their own vision of mathematics, resulting in subtle differences in their vision of rigor, among other elements. Using tools drawn from the anthropological theory of didactics (ATD), and specifically the notion of personal relationship, we provide an analysis of interviews with two students, each of whom was paired with a different teacher interviewed in Hernandes Gomes & González-Martín (2015). Our results show that some elements of the teachers' personal relationship with mathematics also emerge in the students' interviews, in particular those elements pertaining to modelling and estimations.

Introduction

The importance of mathematics and its applications in various scientific and technological fields is undeniable. In engineering, the models required in interpretation and problem solving are usually an adaptation of (or based on) mathematical tools. However, the formalist approach used in mathematics instruction often obscures its practical applications. This "may result in a gap in the students' ability to use mathematics in their engineering practices" (Christensen, 2008, p.131). For instance, Cardella (2013), in analysing how engineering students use mathematical thinking in their capstone projects, noted the following with respect to students' perceptions of precision: "some undergraduate engineering students can become frustrated by the ambiguity and uncertainty that are normal for authentic engineering tasks" (p.96). This remark points to different practices (for mathematicians and engineers), where the 'same' elements acquire different status: for instance, whereas in mathematics, precision needs to be discussed and proved, in engineering it may be taken for granted as an explanation for certain practices.

In this regard, Wake (2014) points out that it is very important "to design curricula in ways that ensure that mathematics is valued by learners as they attempt to make sense of, and with, mathematics in ways that facilitate their being able to engage in practice (doing) and developing their identity (becoming)" (p.288). Regarding this identity, the distinction between mathematics professionals and engineering professionals has long been identified (Snyder, 1912, p.125), and this discrepancy may be present in teaching practices. Engineering students' vision of mathematics, as well as the way they use mathematics in their professional life, is influenced by many factors, including their teacher's approach to the subject. With this in mind, in Hernandes Gomes & González-Martín (2015), we investigated how teachers' academic backgrounds (whether in mathematics, engineering, etc.) shape their vision of mathematics. We saw that the teachers' own exposure to different practices

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seemed to lead them to develop different views on the use of mathematics for engineering (we come back to the main results in the Data section). Consequently, aspects of that research led us to reflect on mathematical practices in engineering courses, specifically. The research presented here focuses on engineering students' vision of mathematics, how it relates to their teacher's vision, and how it is influenced by their teacher's background. We state the objective of this paper at the end of the Theoretical framework section.

Theoretical framework

Given an object *o*, an institution *I*, and a position p in *I*, Chevallard (2003) defines the *institutional relationship with o* in position p as the relationship with the object *o* which should ideally be that of the subjects in position p within *I*. This *institutional relationship* has an effect on the subject, who may belong (or has belonged) to several institutions, where s/he engages in different tasks at the heart of different practices (or *praxeologies*). Every subject x has a *personal relationship* with any object *o* as a result of all the interactions that x can have with the object *o* in different institutions to which x belongs (or has belonged), or in the different positions x can occupy. From this *personal relationship*, an individual will be endowed with what could be designated as 'knowledge', 'know-how', 'conceptions', 'competences', 'mastery', and 'mental images' (Chevallard, 1989, p.227). All these elements are developed by solving specific tasks, using specific techniques that are justified by given explanations (technology). Furthermore, these tasks, along with accepted techniques and explanations, are constrained by the *institutional relationship* that the institution has with the objects at play. Therefore, *institutional relationships* have an effect on an individual's *personal relationship* with an object. This *institutional relationship* depends, among others, on the status assigned to *o* by *I*, which depends highly on p (and therefore also on *I*), or on p's modality of access to *o*, enabled by *I* (Winsløw, 2013). We illustrate this in the next paragraph, using an example.

A student who studies derivatives in a first-year Calculus course in a Faculty of Mathematics is subjected to the relationship $R_M(s, d)$. In subsequent courses, the notion of derivative is revisited and reconstructed in such a way that at the end of the Mathematics program, the same individual is subjected to the relationship $R_M(t, \Delta)$. On the other hand, a student who studies derivatives in a first-year Calculus course in a Faculty of Engineering is subjected to the relationship $R_E(s, \delta)$ and, after using derivatives to solve various engineering-related tasks, at the end of the study programme this individual will be subjected to the relationship $R_E(e, D)$. If these two individuals go on to occupy the position of teacher in a Faculty of Engineering, teaching derivatives, they will both be subjected to the relationship $R_E(t, \delta)$. However, they each will have a different *personal relationship* with derivatives, since they previously occupied different positions in different institutions and therefore probably used the object 'derivative' in different ways. As a consequence, it is possible they do not teach the 'same' notion of derivative to their students. The following diagram shows the different trajectories and *institutional relationships* to which both individuals have been subjected; even if both of them end up occupying the same position in the same institution, their trajectories will have been different:

$$R_M(s,d) \to R_M(t,\Delta) \to R_E(t,\delta)$$

$$R_E(s,\delta) \to R_E(e,D) \to R_E(t,\delta)$$

We therefore believe that the situation of a mathematics teacher in an engineering faculty can be quite complex. The way mathematics topics are introduced in engineering is different than in a mathematics faculty, and an instructor with a background in mathematics should ideally teach engineering students differently than the way she or he was taught mathematics. That said, instructors with a background in mathematics likely have a different teaching approach than engineering instructors who are also professionally active engineers. For the most part, university mathematics teachers (and in particular, those in engineering faculties) have no formal teacher training. Therefore, we conjecture they draw on elements present in their *personal relationship* with mathematical notions and that these elements may influence the vision (and the *personal relationship*) their students develop with respect to mathematics.

Using the tools of the *personal relationship* and *institutional relationship*, this paper's objective is to determine whether Engineering students studying under teachers with different backgrounds consequently develop different *personal relationships* with the mathematical notions they use, and, if so, whether it is possible to identify any common elements between the teachers' and the students' *personal relationships* with specific mathematical notions.

Methodology

It is important to note that the research presented here is in an exploratory stage. To analyse the pertinence of our theoretical approach, we are using already available data. By assessing how the aforementioned theoretical tools can be used to study the type of phenomena we are interested in, we will be able to make decisions regarding methodology and plan an adequate data collection strategy for future research.

Our data are derived from a project that was developed in two stages (Hernandes-Gomes, 2009). During the first stage, interviews were conducted with two teachers (T1 and T2) in an engineering school at a private university in São Paulo, Brazil. T1 is a female teacher with BSc of Mathematics, MSc of Space Engineering and Technology, and PhD in Mechanical Engineering, specialising in neural networks. At the time of the interviews, she had only taught at one university (for 7 years), teaching Differential and Integral Calculus, Analytic Geometry, and Linear Algebra. T2 is a male teacher with a BSc, MSc, and PhD in Mechanical Engineering. T1 usually teaches mathematics courses, whereas T2 normally teaches profession-oriented courses (Introduction to Computational Science, Mechanics of Solids, and Resistance of Materials). Both instructors have overseen student capstone projects, and because this was the only activity they had in common, the interviews focused on their supervision of these projects. The two teachers were interviewed together on two occasions, to encourage dialogue between them and reveal the contrasts between their *personal relationships* with mathematics (Hernandes Gomes & González-Martín, 2015).

During the second stage, interviews were conducted with two students, each of whom was working on a capstone project with a different one of the two teachers from phase one. S1 is a female student finishing her Bachelor degree in Production Engineering. Her capstone project, supervised by T1, concerned artificial neural networks. S2 is a male student finishing his Bachelor degree in Mechanical Engineering. His capstone project, concerning simulation software for mechanical engineering, was supervised by T2. The students' questionnaire was developed following the interviews with their teachers, and included elements pulled from the teachers' responses. The students were interviewed separately in December, 2008. The interviews were audio recorded and transcribed. Figure 1 summarises the profiles of the teachers and students.

Teacher	Gender	Academic Background	Professional background	Student
T1	Female	• Bachelor of Mathematics, • Master of Space Engineering and Technology • Doctorate in Mechanical Engineering	Has taught solely at this one university for seven years, giving courses in Differential and Integral Calculus, Analytic Geometry, and Linear Algebra.	**S1** Female Production Engineering Neural Networks
T2	Male	• Bachelor, Master and Doctorate in Mechanical Engineering	Has taught at the university level for 23 years, and at this university for six years. His courses include Introduction to Computational Science, Mechanics of Solids, and Resistance of Materials.	**S2** Male Mechanical Engineering Simulation Software

Figure 1. Profiles of both teachers and students

Data

In Hernandes Gomes & González-Martín (2015), we discussed the main differences between T1's and T2's approaches to mathematical rigor and approximation, which we summarise briefly here. First, it is important to note that T1 sees herself as an expert in Artificial Intelligence, whereas T2 identified himself as an engineer at different points during the interview (which was not the case with T1). We interpret this as evidence that even though they are both working in the same faculty of Engineering, the teachers see themselves as occupying different positions in different institutions; this is likely because they have participated in different praxeologies, solving different tasks. T1's base training in mathematics is strongly present in her ways of doing. For instance, she uses mathematical tools to check that a result is correct: "What else can be done for certain work, to give it a better foundation, to really be able to say: 'Ah, this work doesn't have any problem?' Oh, let's carry out the statistical analysis of this data". For her, any discourse (or technology) used to verify results usually requires mathematical rigor; results need to be proven mathematically, even when they satisfy the requirements of the constructed models. This way of doing contrasts with T2's practices, which are probably a product of the praxeologies he engages with as an engineer. During the interview, many of T2's statements indicated he justifies some ways of doing based on common engineering practices: "you can do an experimental study"…"which doesn't happen in reality"…"what in fact is an approximation,"…"all of that is experimental. You can also carry out a mathematical simulation, using computer programs, which are professional." During their exchanges, both teachers agreed that the vision of rigor is different for a mathematician than an engineer, which we again interpret as evidence of their experience with different praxeologies in different institutions. In particular, praxeologies in engineering allow for simplifications and approximations in everyday work to simplify calculations (less-than-strict mathematical rigor). Furthermore, T2 also frequently mentioned the use of professional software.

During the student interviews, S1 acknowledged that her profile was more finance-related, and that she had approached T1 to supervise her capstone project because neural networks have applications in finance. She also acknowledged the strong presence of mathematical activity in her project:

> S1: *The tool [T1] had used for her PhD was a probabilistic tool, and so the difficulty lay in adapting the tool to anticipate [stock market] indexes. So, that was the major difficulty, changing the tool – not the type of tool, neural networks, but another type of network that was also new for [T1] [...]. And in addition to that, I also used a classical tool, regression.*

On the other hand, S2 expressed many points of view that seem to indicate his *personal relationship* with mathematics as an engineer is similar in many ways to T2's *personal relationship*. One viewpoint expressed by both T2 and S2 is especially worth noting: neither believes it is worth studying mathematics for its own sake, i.e. without putting it into an engineering context. For instance, S2 was asked which of the results used in his capstone project depended on mathematics. He answered that his theories and formulations stemmed from physics, and that, like all physics theories and formulations, they could be considered mathematical only because physics depends on mathematics: "I can't use physics without mathematics. So, I can consider it as... as mathematics too. But in the end, they are theories and formulations [derived] from physics". We can see that each student experienced totally different praxeologies working under a different teacher, with a different weight given to mathematics:

> S1: *It was like that, doing some tests, using the software and based on the literature to prove the model.*

> S2: *You don't need to use an integral or a differential nowadays. The calculations we do are mostly basic.*

S2 added that while he anticipated using Calculus to prepare his master's, he believed he would never need it in his daily practice as an engineer (in particular, any integral or any differential), stating that the calculations he had to perform were quite basic. His views also echoed T2's vision of approximation, as we discuss below. Not surprisingly, S2 identifies himself as an engineer, while this kind of identification is absent in S1's speech. However, S2 emphasised the need to interpret and understand notions, and insisted an engineer must have a solid vision of physics and a firm grasp of the different magnitudes used. This knowledge is necessary because an engineer often needs to work with estimations and inexact, approximate data:

> S2: *Most engineers nowadays, in my opinion, are trained without a very good physical approach to problems. What does that mean? They sometimes don't know what... what the magnitude of a Newton means, what an interval means, they don't have a concept of the physical unit.*

For S2, theory is a tool that backs up the use of approximations, as revealed in this quote:

> S2: *[regarding the use of software] If you don't understand the physical magnitude of the thing, you'll get totally absurd results, and then he'll look at the software, if he doesn't understand the concept [...]. You need to calibrate the model. Because if you go on refining, refining, refining... [...] you'll get to a size out of its range [...]. And because of the theory, the integrals or differentials used for these elements, the approximation won't be an approximation [but will] become... something else.*

Their responses indicate that the students use mathematical elements in different ways. Furthermore, with respect to software, S2 mentioned different professional engineering programs (indicating his proficiency with them) and said that computers are necessary tools for engineers, reflecting the answers of T2. He stated that less-experienced engineers risk applying mathematical models strictly, leading them to wrong interpretations. S2 also said that some engineering problems cannot be solved by hand (another issue raised by T2), and that it takes a degree of intuition to know when to initiate computer modelling. On the other hand, S1 used Matlab and Minitab for the statistical analyses.

Coming back to the different ways the students described their activity, the following quotes illustrate the students' visions of their capstone project:

> S1: *One of the difficult aspects was that [...] in neural networks, you can use as much data as you wish, it even makes the model better. In regression, you can't do that. Rather, you have to pay attention to the right amount [...]. So, in the modelling, I had to pay attention, work with certain data [...].*

> S2: *In engineering [...] you always work with tolerance [...] and it has to be below a given limit. So... the result doesn't matter, it's... you test one piece, and there you get a given* x*: you don't need to get exactly to that x. You have to make a calculation that gives less than* x*. And there, it's good, it's approved [...]. I think that's what engineering is. At least, in my area of work, it's been like that.*

S1's description of her work seems to portray the activity of a mathematician, and the application of mathematical models is very present. On the other hand, S2 directly addressed his vision of engineering, and it is clear that he has been involved in praxeologies where approximation and tolerance are quite frequent. This recalls Cardella's statement, mentioned in our introduction, about the normality of ambiguity and uncertainty in authentic engineering tasks (2013, p.96), and contrasts strongly with the role of approximation in S1's work:

> S1: *Regarding approximation, I remember... we had some teachers in the introductory courses, they insisted a lot on that: that we could use approximate models, but with several decimal figures. And paying attention. Nowadays, professionally, I'm working a lot with accounting, and there you can't approximate anything: it either is or it isn't. So, that's it, there's no such approximation: you need to have the exact value.*

Many of S1's opinions hewed closely to statements made by T1. Regarding approximation, S1 and T1 both asserted that results obtained by engineers need to be analysed and checked using a variety of tools. S1 pointed to one of her results which allowed her to use neuronal networks to predict stock market data with 98.6% accuracy, and discussed the importance of using approximations. An interesting point of divergence between S1 and T1 concerned the importance of studying pure mathematics. T1 expressed the opinion that an engineer must acquire basic mathematical tools. She also stated that she shows students in her mathematics courses how these tools will be applied in their professional practice. However, S1 stated that the models used in these courses are always ‘perfect’ and that the mathematics found in textbooks is very difficult to apply in practice, using real data. This supports Christensen's work (2008), which identifies a gap between mathematical tools and engineering practice. S1’s belief, also expressed by some researchers, is that “Mathematics learnt in the faculty have little or nothing to do with the real world”, which seems to indicate that her active engineering practice may influence her own *personal relationship* with mathematics in ways that diverge with T1’s *personal relationship*.

Regarding this point, S2 also expressed concerns that resemble those of T2: when asked how his courses could be changed to facilitate the completion of capstone projects, S2 pointed out the need to show the practical application of results, i.e., how an engineer would use results in her or his daily work. He also expressed his belief in the importance of proving these results. In this respect, T2’s and S2’s *personal relationships* with proof seem very similar: they both agree that an engineer needs to know what an integral is or how to solve a differential equation, even if this knowledge is not used in an engineer’s daily practice. S2 also mentioned the need for practical examples. This may be because he occupies a different position than T2: T2 is an experienced engineer, familiar with the application of mathematics in engineering, whereas S2 is still a student who does not yet fully understand how and when these skills will be needed in his daily practice. Regarding the use of mathematics, S2 saw his own case as rather atypical: although he had used finite elements in his capstone project and was applying Calculus as he worked towards his master's degree, he said many of his fellow students continued to question the requirement to study certain mathematical results.

Final considerations

This exploratory work looks at common and divergent visions of mathematics espoused by engineering students and their teachers, and the adequacy of the notion of *personal relationship* to study these visions. Our data seem to indicate that the visions of S1 and S2 differ, and that although they both were completing engineering capstone projects, they were actually engaged in very different praxeologies. S1, supervised by a teacher with a background in mathematics, appears to have developed a vision that stresses rigor, and her approach to engineering is influenced by elements of mathematical praxeologies: analysing data, constructing models, proving their efficacy using statistical regression, etc. At the same time, her engineering experience seems to influence her *personal relationship* and she complains that some of the models studied in her courses are always ‘perfect’. S2, however, seems to have developed a *personal relationship* that is much closer to T2's, who has training and experience as an engineer. Both S2 and T2 seem engaged in praxeologies that are more closely related to an engineer’s practice, using approximations and interpretations and employing computer software for testing. Whereas S1 seems more critical of the role of mathematics courses in her education, S2 seems to have developed a somewhat contradictory opinion, stating that it is necessary for engineers to understand mathematics, although his daily work as an engineer will not require him to apply this knowledge.

This preliminary work indicates that further investigation is needed to explore the complex phenomenon of teachers with various backgrounds educating future engineers, and how their different *personal relationships* influence their students' vision and use of mathematics. The tools provided by ATD seem to provide a way to identify implicit differences in teachers' views and practices, and reveal how these practices (or praxeologies) influence their students' work.

However, we are aware of two main biases in the results presented here. First, our participants are rather 'extreme' cases, making it easier to pinpoint these differences — we are analysing individuals engaged in very different praxeologies. Taking this into account, we plan to apply our tools to more homogeneous populations and practices; for instance, teachers giving the same first-year Calculus course. This will allow us to determine whether 'an identical' praxeology is developed the same way by teachers with different backgrounds. Secondly, we are also aware that the data analysed here were collected without specifically taking into account the framework provided by ATD. Given that these tools seem to help identify certain interesting elements in teachers' and students' responses that could be linked to institutional elements, we plan to continue our research by constructing a suitable methodological dispositive, with a sample that shares more commonalities.

The next step of our research entails working with six teachers of a first-year Calculus course in engineering, each with a different background. We intend to analyse how these instructors develop praxeologies for teaching Calculus, how these praxeologies differ, and how they relate to the teachers' different backgrounds and their *personal relationships* with the content of their Calculus course. The results of our analyses will be the source of future papers.

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Conceptualizing students' processes of solving a typical problem in the course "Principles of electrical engineering" requiring higher mathematical methods

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The project KoM@ING aims at studying the mathematical skills required in technical subjects of bachelor programs in engineering. Our subproject is especially interested in the first-year-course "principles of electrical engineering". We will present our analysis of an exercise dealing with the time-dependent behavior of a quantity in an oscillating circuit, which can be described using ordinary differential equations of degree one and two. The presentation contains the newly invented concept of the student-expert-solution, which is a normative solution of the task and used as a basis for our empirical studies. We analyze students' problem solving processes as well as written exams concerning the same task. We will point out specific difficulties and challenges.

Introduction

We are interested, which mathematics is used in basic courses on electrical engineering in electrical engineering study programs, how mathematics is used and how the way mathematics is done differs from inner-mathematical contexts. To answer these questions, we analyze how first year students of electrical engineering solve electrical engineering tasks, which require knowledge and cognitive recourses from both mathematics (school, university) and electrical engineering. The mathematical practices in mathematical contexts look different from those in engineering contexts (see Redish, 1995). We try to find out the difficulties and the challenges caused by these differences.

In German universities engineering students typically have to take courses in engineering subjects like the "Fundamentals of Electrical Engineering" (FoEE) and courses on the "Math for Engineering Students" (MfES) in the same semester. In FoEE-courses the theory is presented in the lecture while the accompanying exercise classes show ways to solve problems by simplifying the content of the lecture and making it applicable. In written examinations the students often do not need justifications for their solutions to get points. On the other hand, the MfES-courses show calculation methods rather than the proving of mathematical statements like in math lectures for students of mathematics. The content of the MfES-course is calculus in one and in higher dimensions, linear algebra (solving of lineare equation systems, theory of eigenvalues) and complex numbers.

The separation between the two subjects leads to some challenges for students: There are asynchronisms between lectures on MfES and FoEE, i. e., often a mathematical topic is needed in FoEE before it is presented in MfES. As lectures on mathematics have a deductive structure to assure understanding, it is not possible to adjust the order of the mathematical

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topics in the MfES-course in every case. There is also a different mathematical practice in MfES and FoEE, e. g., in the use of vectors and differentials.

We have the following research questions: (1) Which (idealised) solutions can we expect from students after their second semester of their electrical engineering studies? (2) How do students actually solve exercises in FoEE-courses and which difficulties do occur? (3) Which solving strategies do students use in their solution processes?

We focus on four tasks of the second part of the “foundations of electrical engineering”-course (the so-called GET-B-course), which students are to take in their second semester. All the students’ written work in an exam containing the four exercises was scanned and the same tasks were given to eighteen pairs of students, whose work and communication were video-recorded and transcribed.

Theoretical Background and Methodology

Our analysis of the problem solving processes is based on two tools:

At first we need to describe the competences required in the task solving in engineering subjects. The so-called student-expert-solution (SES) is a “normative solution” that is based on the modeling cycle by Blum/Leiss, 2005, and the description of problem solving processes by Polya (1949). Another foundation are solutions of the tasks done by electrical engineering experts in an expert interview, who were asked to solve the tasks from the perspective of a first-year student who well understood the contents of the GET-B-course. The SES is complemented with related theory-based comments in a second column, forming the so-called TESES, the theoretically enhanced student-expert-solution. The TESES is used to sharpen the theoretical description of the tasks and to analyze students’ solving processes.

The starting point of our analyses was the “model solutions” on the tasks which were provided for the correction of the written exam. But there was the problem, that the model solutions of the exercises on electrical engineering mostly just contain the calculations or sometimes even just the result. In order to get detailed and normative solutions of the exercises we conducted interviews with the task designer and other electrical engineering experts using the so-called PARI-method (Hall et al., 1995), which is a task-based interview-technique. The abbreviation PARI stands for Precursor, Action, Result and Interpretation. Using this kind of interviews, the normative solution could be supplemented with competence expectations and additional remarks.

The interview consists of three phases: the first step is the solving of the exercise without interruptions. Then a reconstruction of the reasons for the way the exercise was solved is done in order to identify the used resources and to get justifications for each step of the solution. Finally, there is a didactic reconstruction of the exercise. This reconstruction consists of the scrutinizing of alternative solutions to the exercise, typical mistakes of students after their first year and possibilities for the validation of the results. In a second step, the expert is asked for reasons for assigning the exercise and possible variations for exercises on this topic.

On the other hand, we base our analyses on theory of modelling and problem solving. The modelling cycle (Blum/Leiss, 2005) divides the solving of modelling tasks into seven steps:

(1) understanding of the task and construction of the "situation model", (2) simplifying and structuring of the situation: construction of the so-called "real model", (3) translating into a mathematical problem, (4) mathematical work, (5) interpretation of the result in the real world, (6) validation and (7) presenting of the results. It divides the process into "reality" (step 1, 2, 6 and 7) and "mathematics" (step 4); step 3 and 5 connect the two parts of the cycle. On the other hand, we use a conceptualization of mathematical problem solving processes and heuristics by Polya, 1949, that describes four phases: the understanding of the problem, the devising of a plan, the carrying out of the plan and looking back.

Our second tool is the so-called low-inferent analysis (LIA), which is used to analyze the solving processes of the student pairs. The LIA has four functions: at first, it connects the phases in the SES and the phases shown in the students' solving processes. It describes the differences between the idealized solution paths in the SES and the actual solving processes. We annotate and interpret the differences to describe strategies of the students, which are independent from the actual exercises. At last, the LIA is used to find connections to the epistemic games resp. e-games (see Tuminaro/Redish, 2007) and justification strategies (see Bing, 2008). The e-games frame videographed solving processes in physics by three framings: quantitative sense-making, qualitative sense-making and rote equation chasing. The e-games were advanced by conceptualizations of mathematical justifications in physics: Calculation, Physical Mapping, Invoking Authority and Math Consistency. For example, "calculation" stands for "a correct completion of an algorithm gives a correct result".

We also scanned 92 solutions of the four tasks that were done by students in a written examination to the second part of the "foundations of electrical engineering"-course. We divided each subtask into different steps that would have to be done in order to solve it. We categorized each activity in a way, which is independent from the original marking of the examination and that consists of three degrees: the student gets a 2, if their solution was totally correct. They get a 1, if there were faults but their solution still contains right parts. The student got a 0, if their solution was totally wrong. If a previous step was wrong (coded as 0) or contains faults (coded as 1), but the student does the subsequent step in the right way, they get a 2 in the subsequent step.

The following diagram shows how the different tools are used in our analysis:

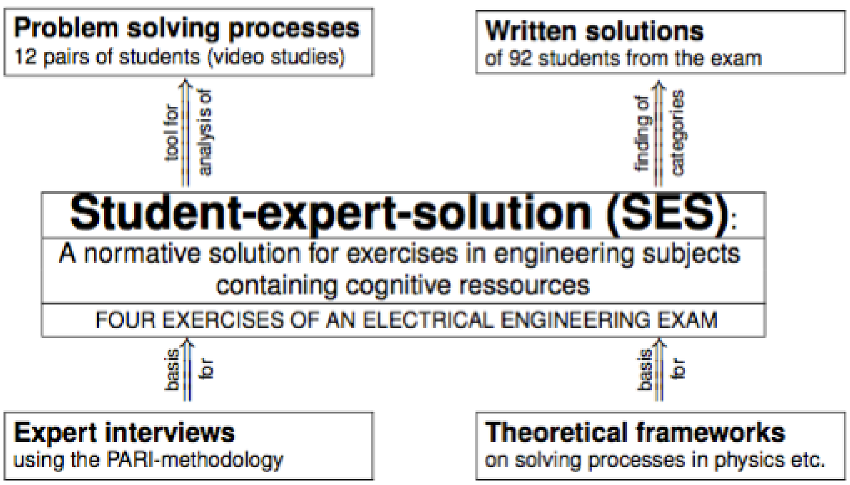

Fig. 1: Diagram on the connection of the different elements of our analyses

A Short Outline of the Exercise and Its Solution

We now present one of the four exercises, the so-called $A2$, which deals with transients in an oscillating circuit containing a resistor R, an inductor L, a capacitor C and an ideal voltage source U_0. It is shortened for brevity. $A2$ starts with the following sketch of the circuit:

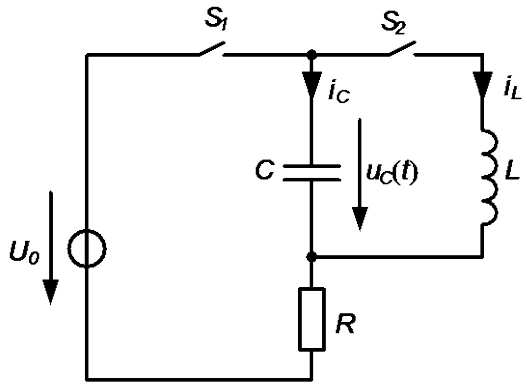

Fig. 2: Sketch of an oscillating circuit containing two switches, an ideal voltage source, a resistor, a capacitor and an inductor

At the beginning the switches S_1 and S_2 are open and the inductor and the capacitor are totally discharged. At the moment t = 0 the switch S_1 is closed, while S_2 remains open.

In subtask 2.1 and 2.2 the students are to give the values of $u_C(t)$, the voltage at the capacitor, $i_C(t)$, the electric current in the capacitor, and $i_L(t)$, the voltage at the inductor, before and after the opening of S_1. Solution: All three values are 0 before S_1 is closed, because the components of the circuit are totally discharged. After closing the switch $u_C(t)$ and $i_L(t)$ are still 0, as a voltage at a capacitor resp. an electric current at an inductor does not change discontinuously; a fact the students learn at the lecture. $i_C(t)$ equals U_0/R using Ohm's law.

In subtask 2.3 the students are to form an ordinary differential equation for $u_C(t)$. Solution: We have to apply Kirchhoff's voltage law on the left part of the circuit, giving $U_0 = u_C(t) + u_R(t)$, and use the two component equations of the capacitor $C\dot{u}_C(t) = i_C(t)$ and the resistor $u_R(t) = i_C(t)R$. The combination of those equations gives an ordinary differential equation (ODE) of first order, which is $u_C(t) + RC\dot{u}_C(t) = U_0$.

This differential equation is to be solved in subtask 2.4. Solution: The solution can be done using on the one hand the separation of variables combined with a variation of constants. On the other hand the solution can be found by superposition of the solution of the homogenized ordinary differential equation, one particular solution of the inhomogeneous ODE and the using of the initial value $u_C(0) = 0$. The solution is $u_C(t) = U_0(1 - e^{t/(RC)})$.

In subtask 2.5 the students are to sketch the voltage curve of $u_C(t)$ and $i_C(t)$. Solution: The graph of $u_C(t)$ starting at $u_C(t = 0) = 0$ approaches an asymptote at $u_C(t) = U_0$, because $e^{t/(RC)}$ converges to 0 for $t \to \infty$. We get the function for $i_C(t)$ by another combination of the formulas from 2.3 and inserting of the solution for $u_C(t)$ from 2.4.

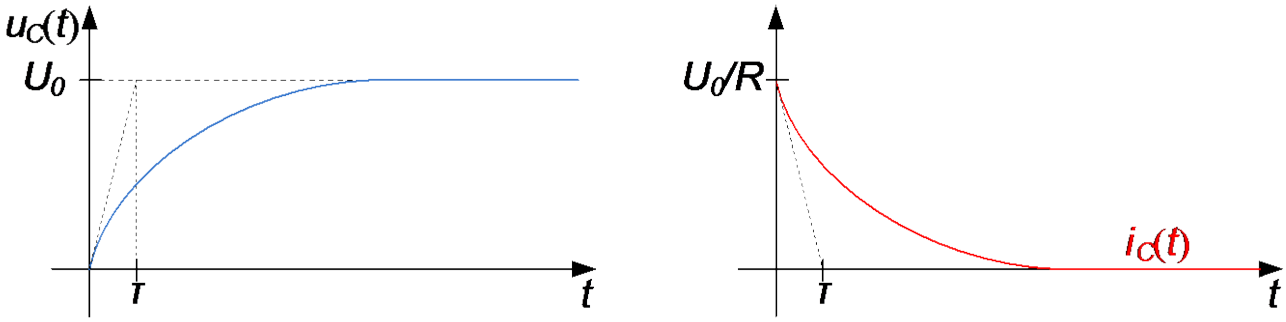

Fig. 3: Graph of $u_C(t)$ and $i_C(t)$ containing the time constant τ

Before subtask 2.6, the transient after 2.5 is defined as completed, i. e. $u_C(t)$ is set to be equal to U_0. Then S_1 is opened and S_2 is closed. So, in the first step the students have to translate the components of the following circuit and its experimental set-up into equations:

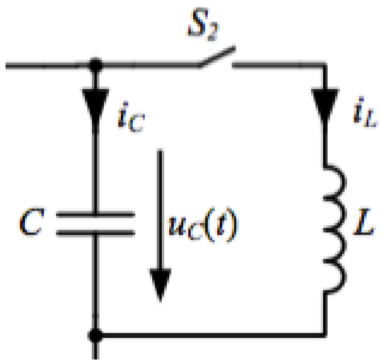

Fig. 4: Sketch of the right part of the oscillating circuit

This second part of the exercise is analogous to the first part, i. e., the students at first have to find the values for $u_C(t)$ and $i_C(t)$ in the changed situation, they have to find component equation and apply Kirchhoff's laws and in the last step, they have to form and solve an ordinary differential equation. In subtask 2.6 the students have to find the values for $u_C(t)$ and $i_C(t)$ before and after the switching of S_2 and justify their solutions. Solution: Before and after the switching, we get $u_C(t) = U_0$ and $i_C(t) = 0$, as the transient is completed and the capacitor is fully loaded. The values do not change discontinuously, because of the component equations of the capacitor resp. the inductor, which gives a continuous function.

In subtask 2.7, the forming of the differential equation, the students have to find and combine four formulas. They need the component equations for the capacitor $(C\dot{u}_C(t) = i_C(t))$ and the inductor $(L\dot{i}_L(t) = u\_L(t))$, as the circuit contains these two components. The application of Kirchhoff's laws gives $u_C(t) = u_L(t)$ and $-i_C(t) = i_L(t)$. The equations can be combined by insertion of the first derivative of the component equation of the capacitor, which leads to an ordinary differential equation of degree two: $LC\ddot{u}(t) + u(t) = 0$.

This differential equation is to be solved in subtask 2.8. The students learn approaches to solve such equation in their MfES-courses. We set $u_C(t) = e^{at}$, which leads to the characteristic equation $LCa^2 + 1 = 0$, which only has complex solutions. The MfES-courses give the solving approach $u_C(t) = A \cdot cos(at) + B \cdot sin(at)$. Using the initial values for $u_C(t)$ from part 2.6 gives the solution: $u_C(t) = U_0 \cdot cos(at)$. The result can be validated by the consideration of the underlying physical situation: As the value of the voltage oscillates, it has to be described by trigonometric functions and cannot be solved by an exponential function as in subtask 2.4.

Preliminary results of analyzing students' solutions using the SES

This exercise shows various characteristics of the use of mathematics in engineering subjects. The sketch of the circuit is a conventionalized help for the mathematization of the given situation. As stressed in Biehler et al., in press, the students do not need a real model as suggested in the modeling cycle, but they need methods to understand the given conventionalized sketches, in this case of the oscillating circuit. In contrast to the exercise on the magnetic circuit, in which the students have to translate a sketch of a magnetic circuit into an equivalent circuit diagram, in this task the sketch in figure 1 can directly be mathematized to find equations needed for the ordinary differential equation.

In the transition from 2.1 to 2.2, i. e. the switching S_1, the students argued in two different ways: on the one hand they used the physical arguments shown above in the SES to this exercise, mainly saying, that the two values ($u_C(t)$ and $i_L(t)$) do not change discontinuously, as they learned in the GET-B-lecture or experiments in lab classes at university.

From a mathematical point of view other students got the same results using the component equations of the capacitor and the resistor. $C\dot{u}_C(t) = i_C(t)$, the component equation of the capacitor, shows, that the current at a capacitor is dependent on the change of voltage. This means, that $u_C(t)$ can be computed as an integral with integrand $i_C(t)/C$ and as the integral of a continuous function is a continuous function, $u_C(t)$ cannot have points of discontinuity.

The application of Kirchhoff's laws can be seen mathematically as an application of graph theory. Applying Kirchhoff's voltage law, the students have to find so-called meshes, which are directed, valued and closed paths in graphs. In this case the values are the different voltages in the mesh. Kirchhoff's current law, the second law of Kirchhoff, also has a graph theoretical interpretation using nodes.

In subtask 2.3 the students have to use "equation management" (see Biehler et al., in press), which consists of two steps: First the students have to recall the relevant formulas (in this case, the component equations and the equations formed using Kirchhoff's laws) and then to transform these equations to get an ordinary differential equation for $u_C(t)$ only containing $u_C(t)$, its derivatives and constants. This extends the concept of equation management as the equations contain functional expressions like $u_C(t)$ or $i_C(t)$, derivatives and the students are forced to use analytic methods like derivation and integration to combine equations.

The solving of $u_C(t) + RC\dot{u}_C(t) = U_0$ is part of the "world of mathematics" as described in the modeling cycle as the students do not have to take the real situation into account in this step. $u_C(t)$ can totally be solved using methods and facts learned in MfES-courses. In a first step, the students have to solve the homogenized ordinary differential equation, in which all terms without $u_C(t)$ or its derivatives are set as 0. The solution of the homogenized ordinary differential equations, where U_0 is set as 0, must be an exponential function as the first derivative of $u_C(t)$ is a multiple of $u_C(t)$. A particular solution of the inhomogeneous ODE can be found as $u_C(t) = U_0$, as the derivative of a constant is 0. Both experts and students argued this way before solving the task.

When the students sketch the graph of $u_C(t)$ in subtask 2.5, they can either use their solution of 2.4 or the physical mechanisms (see Tuminaro/Redish, 2007, resp. Bing, 2008) knowing that in a circuit containing a resistor, a capacitor and an ideal voltage source, the capacitor loads up until it reaches the value of the ideal voltage source, in this case U_0. The students get to know this behavior in experiments done in lab courses. Therefore, the graph of $u_C(t) = U_0(1 - e^{-t/RC})$ approaches U_0, which is an asymptote of the function $u_C(t)$.

The following subtasks of the exercise are analogous to the recently presented subtasks. Subtask 2.6 can be solved by looking at the physical mechanisms of such a experimental set-up or by interpretation of the component equations of the capacitor resp. the inductor. In 2.7 the students have to find the required formulas by mathematization of the components using a translation of each component to its component equation as well as applying Kirchhoff's rules on meshes and nodes, i. e., they have to use graph theory again. In this case, the equation management also includes methods of calculus, which leads to an ordinary differential equation of degree two. This can be solved using methods taught in the lectures on MfES, so again the "world of mathematics" is entered. The validation of the results can be done by again looking at the physical mechanisms. As the value of $u_C(t)$ oscillates, it has to be described using combinations of trigonometric function instead of an exponential function.

Summary and Conclusions

The student-expert-solution (SES) is a newly invented tool to describe problem solving processes in the field of basic engineering courses like the Fundamentals of Electrical Engineering or Fundamentals of Mechanics for mechanical engineers resp. civil engineers. It is a detailed and normative solution for engineering tasks, which is complemented by theory-based comments and remarks of experts concerning competence expectations and additional remarks on the certain type of exercise like typical mistakes, variations or reasons for the assigning. The SES conceptualizes the solving process in a general way and divides it into three phases: mathematization, math-electrotechnical working and validation.

The phases have special characteristics in electrical engineering in the first two semesters:

The mathematization-part typically contains the use of conventionalized sketches (like sketches of circuits or equivalent circuit diagrams), instead of constructing a „real model" (as suggested in the modelling cycle) or drawing a figure (see Polya). In a second step, students have to apply certain rules for the translation of the components and the experimental set-up in order to get a task, which is solvable using methods learned in MfES and FoEE.

The math-electrotechnical working contains a working with quantities and using resources like equation management, which are not solely based on pure mathematics. The students do not directly apply calculation methods learned in the MfES-courses, which would be part of a "world of mathematics".

The validation consists of an analysis of units and magnitudes or the comparison with behavior known from experiments in lab courses. The students do not do a validation of the model adequacy, but they validate the used mathematical approach.

Outlook

Our analyses will be extended by the analyses of the work of the students in the remaining exercises of the GET-B-exam, which require various kinds of mathematics. In one exercise on magnetic circuits, the students just need to form and combine equations to calculate physical quantities (see Biehler et al., in press). In another exercise they need to solve integrals in one variable to calculate for example the root-mean-square-value or the rectified value, which can be directly translated into the calculation of certain means. In the last exercise on complex alternating current, students have to calculate quantities using complex numbers and so-called vector diagrams. This approach uses an isomorphism between trigonometric functions, which describe the values of voltages and currents in alternating currents, and the complex exponential function in the complex plane.

Acknowledgements

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Interactive tools in lectures with many participants

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On this poster I present different possibilities for the activation of students in lectures with many participants including an instant feedback for both sides on the learning progress. I examine a paper-based evaluation tool (EvaExam) and different online voting tools (Eduvote and arsnova.net). Advantages and disadvantages are discussed.

Introduction

The first year course "Mathematics for economy students 1 and 2" at Leibniz Universität Hannover is held in a group of approximately 600 students. It consists of a lecture (two hours), a central tutorial (two hours) and small tutorials (two hours) weekly. Especially in large groups it is difficult to get direct and immediate feedback from all students about their understanding and their learning progress. Usually only a small amount of students participates actively and answers my questions. A lot of students are quite passive and just listen and copy the contents from the board during the lecture.

To cope with these problems I tried different evaluation and voting tools to increase the interactivity in my lectures and to reach the following aims:

1. Getting direct information about the actual knowledge from most of the students and not only from some people who participate actively. All students should get the possibility to participate anonymously.
2. Increasing interactivity during the lesson
3. Motivating the students to think and work on their own or together with a seatmate
4. Increasing attention during the lesson due to small active breaks

As in the TV show "Who wants to be a millionaire" it is possible to use so-called clickers which are small voting computers which have to be handed out to the audience. But due to the size of the group the use of those clickers was not suitable. So I started to use different tools presented in the next paragraphs.

Use of the paper-based evaluation tool EvaExam

Firstly I tried the paper-based evaluation tool EvaExam. In the beginning of the lesson a questionnaire was distributed among the students. During the lecture I presented six multiple and single-choice questions and the students had to mark the correct answers within some minutes. Finally the questionnaire was collected, scanned and automatically evaluated. One week later the solutions to the exercises were discussed during the lecture. It turned out that the personal and time costs are too high and that the students are dissatis-

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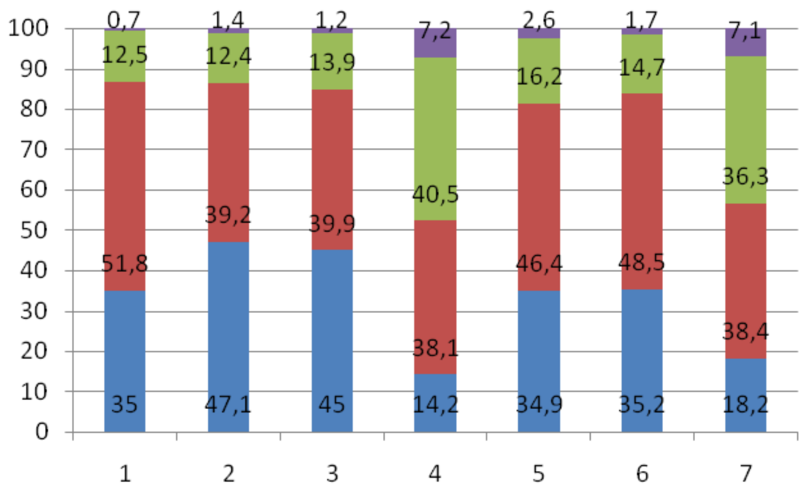
fied with not getting their results immediately but one week later. Nevertheless participants valued the use of multiple-choice questions.

Evaluation of the use of multiple-choice questions (n = 430):

The exercises …
1. … help me to check if I understood the lesson.
2. … help Dr. Leydecker to find out where students have problems.
3. … help Dr. Leydecker to find out which part of the lesson should be explained again or in a more detailed way.
4. … help me to keep concentration during the lesson.
5. … loosened up the lesson.
6. The number of exercises is appropriate.
7. The time to solve the exercises was appropriate.



full agreement (blue column), agreement (red column), rather no agreement (green column), no agreement (violet column)

Use of online voting tools

There are different tools to get direct feedback from the students with less effort. On the one hand I used the commercial tool EduVote (http://www.eduvote.de/) which has to be downloaded onto the smartphone or the tablet computer from the app store. On the other hand there is the web-browser based tool ARSnova (https://arsnova.eu/mobile/). Both programs have in common that the students can enter their choice anonymously into their smartphones and the lecturer gets a direct feedback which can also be projected using the data projector during the lecture. Using EduVote every computer can enter a result twice to give people without a smartphone or flatrate the possibility to enter their results into their seatmates' mobile devices.

As a test I used EduVote during a repetition lesson with different linear algebra and analysis questions to prepare the students for the upcoming examinations. The questions required active computations from the students.

When they were shown the questions the students started to calculate on their own but also discussed the questions with their seatmates which is also desired. In the auditorium one could observe a "productive whisper". When the voting was stopped the students became quiet and waited for the results. Finally the result was explained in detail by a voluntary student or by myself.

During the tests it turned out that it takes a lot of time to wait for the answers of all students as the group's performance and learning speed varies substantially. Therefore I usually stopped the evaluation at approximately 220 answers (more than 50% of the audience) to have enough time for discussing the result and for further explanations.

Evaluation of the use of EduVote (yes/no):

	n=220 (+/-20)
I regard it as a problem that not all students have a smartphone with a flatrate	15%
I have got a flatrate for my device	68%
The interactive exercises during the lecture are helpful to learn the material	72%
Interactive questions during the lessons are positive	92%
I would prefer to have interactive questions more often	93%

Conclusion

The use of interactive elements has several advantages:

- Large participation of the students, every student can participate without having the fear of being embarrassed.
- The students get an immediate feedback about their knowledge and the knowledge of the whole group.
- The students can communicate with each other about mathematics during the lesson.
- The attention during the lecture is increased.
- The lecturer gets an immediate feedback and can eventually make her/his explanations clearer.
- But there is a big disadvantage: One question takes about five minutes of time so that it is not possible to use the program too often.

Acknowledgements

I would like to thank the eLearning Service team at Leibniz Universität Hannover, especially Gabi Diercks-O'Brien and Stefan Frohwein.

Applying an extended praxeological ATD-Model for analyzing different mathematical discourses in higher engineering courses

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In this presentation an extended praxeological model based on concepts from the Anthropological Theory of Didactics (ATD) is used to analyze the relationship between different mathematical discourses in advanced engineering courses, such as Signals and System Theory (SST). The model allows in particular to discriminate between practices and reasoning patterns established in HM-courses and in SST-courses. Its usefulness is exemplarily illustrated by the analysis of a sample solution to a problem from a SST-course provided by the lecturer.

Introduction and research question

Engineering students encounter mathematical practices and reasoning patterns both in courses for Higher Mathematics (HM) and in advanced engineering courses, such as Signals and System Theory (SST). The mathematical discourse in SST-courses includes HM-practices, combines HM practices with electrotechnical rationales and constructs new mathematical practices provided with specific electrotechnical reasoning patterns. The basic analytical tool, to analyze the relationship between the different mathematical discourses, is an extended praxeological model based on concepts from the Anthropological Theory of Didactics. The extended praxeological model allows in particular discriminating between practices and reasoning patterns established in HM-courses and in SST-courses (Hochmuth & Schreiber, 2015a, b).

In this presentation we focus on an SST-exercise and the sample solution given by the lecturer. The guiding research question is: What praxeologies arise in the sample solutions and how are the different mathematical concepts, established in HM- and SST-courses, related?

The extended praxeological ATD-Model

In ATD (Chevallard, 1992, 1999; Winsløw, Barquero, Vleeschouwer & Hardy, 2014) the classic praxeological model is the so called "$4T$-model $[T, \tau, \theta, \Theta]$", where T is the type of task, τ are the appropriate techniques to solve the task, θ are technologies explaining and justifying the techniques and Θ denotes the underlying theories justifying the technologies. This model is applicable to every human activity.

In order to analyze the praxeological structure of higher engineering courses and especially focusing on the complex mathematical discourse of SST-courses, we extend the classical $4T$-model in the following way

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$$\left[T^*, \frac{\tau_{HM}}{\tau^*_{SST}}, \frac{\theta_{HM}}{\theta^*_{SST}}, \Theta^*_{SST}\right]$$

Techniques and technologies are differentiated in two branches: To discriminate between the two branches we focus on technological aspects: If a technique is motivated, explained or justified by electrotechnical or physical reasoning, it is labeled as a SST-technique (τ^*_{SST}). The corresponding technologies are denoted by SST-technologies (θ^*_{SST}). Those techniques and technologies are in a sense related to real world phenomena. If techniques and technologies are related to mathematical concepts established in the HM-courses (e.g. "because the function is continuous"), they are seen as HM-techniques (τ_{HM}) and -technologies (θ_{HM}).

The * denotes the result of a didactic transposition process (Chevallard, 1991; Castela, 2015). Although HM-courses are also results of didactic transposition processes, we omitted the * in the HM-branch, because the didactical transposition of HM-courses takes place in other institutions than the didactic transposition of SST-courses. In this way, we indicate our focus on lecture notes and course materials from higher engineering courses.

A praxeological analysis of a sample solution to a SST-problem

This section shows the application of the extended praxeological model to an exercise from a problem set of a SST-course[1]. The exercise is given as follows:

Assuming $0 < m < 1$, thus $A(t) > 0$, (the envelope of an AM-signal is always positive), show that the above-mentioned envelope detector actually delivers a signal proportional to $A(t)$.

This exercise is also part of the lecture notes and is assigned directly after the introduction of the envelope detector as part of the section on amplitude modulation (AM). The sample solution given by the lecturer is shown in the appendix.

In the following, we illustrate the solution step by step:

- Application of the envelope detector
- Application of the envelope detector requires rectification and application of the low pass filter
- Using absolute value to rectify the signal
- Expanding the rectified signal into a Fourier series to decompose the signal into a constant component and oscillating components
- Applying the low pass filter to get rid of the oscillating components (carrier signal)
- Reading off the proportionality

[1] The authors are grateful to Prof. Dahlhaus (University of Kassel) for placing the exercise and sample solution at our disposal.

Praxeological analysis of the sample solution

The solution steps are now considered under ATD-aspects and assigned to the different branches of the extended praxeological model: The task T^* is to show that the envelope detector delivers a signal proportional to the amplitude $A(t)$ of the modulating signal. To solve the task, it is necessary to apply the envelope detector (τ^*_{SST}) and then to read off the proportionality (τ_{HM}). Elements of technology are, that the modulation index m fulfills the necessary condition for the application of the envelope detector (θ^*_{SST}) and, that electrotechnical quantities are construable as variables in the context of linear functions (θ_{HM}). The application of the envelope detector requires first to rectify the signal (τ^*_{SST}) and second to apply the low pass filter (τ^*_{SST}). Both techniques are essential for the reconstruction of the modulating signal (θ^*_{SST}). The low pass filter suppresses the frequencies of the carrier signal (θ^*_{SST}). The rectification of the signal is done by taking the absolute value (τ^*_{SST}). This technique is classified as an SST-technique, because of taking the absolute value of a signal is the mathematization of rectification (θ^*_{SST}). Therefore, taking the absolute value is motivated by electrotechnical reasoning. For applying the low pass filter the constant component of the signal has to be separated from the oscillating components (τ^*_{SST}) and then, the oscillating component is omitted (τ^*_{SST}). To drop the oscillating components of the signal is a mathematical model for the action of a low pass filter (θ^*_{SST}). The decomposition of the signal can be done via Fourier series expansion (τ_{HM}), because the signal is described as a continuous and periodic function (θ_{HM}). The Fourier series expansion demands ambitious mathematical techniques (τ_{HM}) and technologies (θ_{HM}), for example manipulating infinite sums due to symmetry arguments. After the full Fourier series expansion of the signal is done, the first three summands are written down and finally, the oscillating summands are omitted due to application of the low pass filter. The remaining constant component of the signal is proportional to the amplitude of the modulating signal.

A graphical visualization of this analysis is shown in Figure . The graph shows both the structure of the sample solution and the result of the praxeological analysis. It is constructed as follows: if techniques require complex activities they become subtasks on a lower level. In the next step the praxeological model is applied to each vertex. To highlight the two branches we used different colors. The HM branch is colored in dark blue, the SST-branch in light blue.

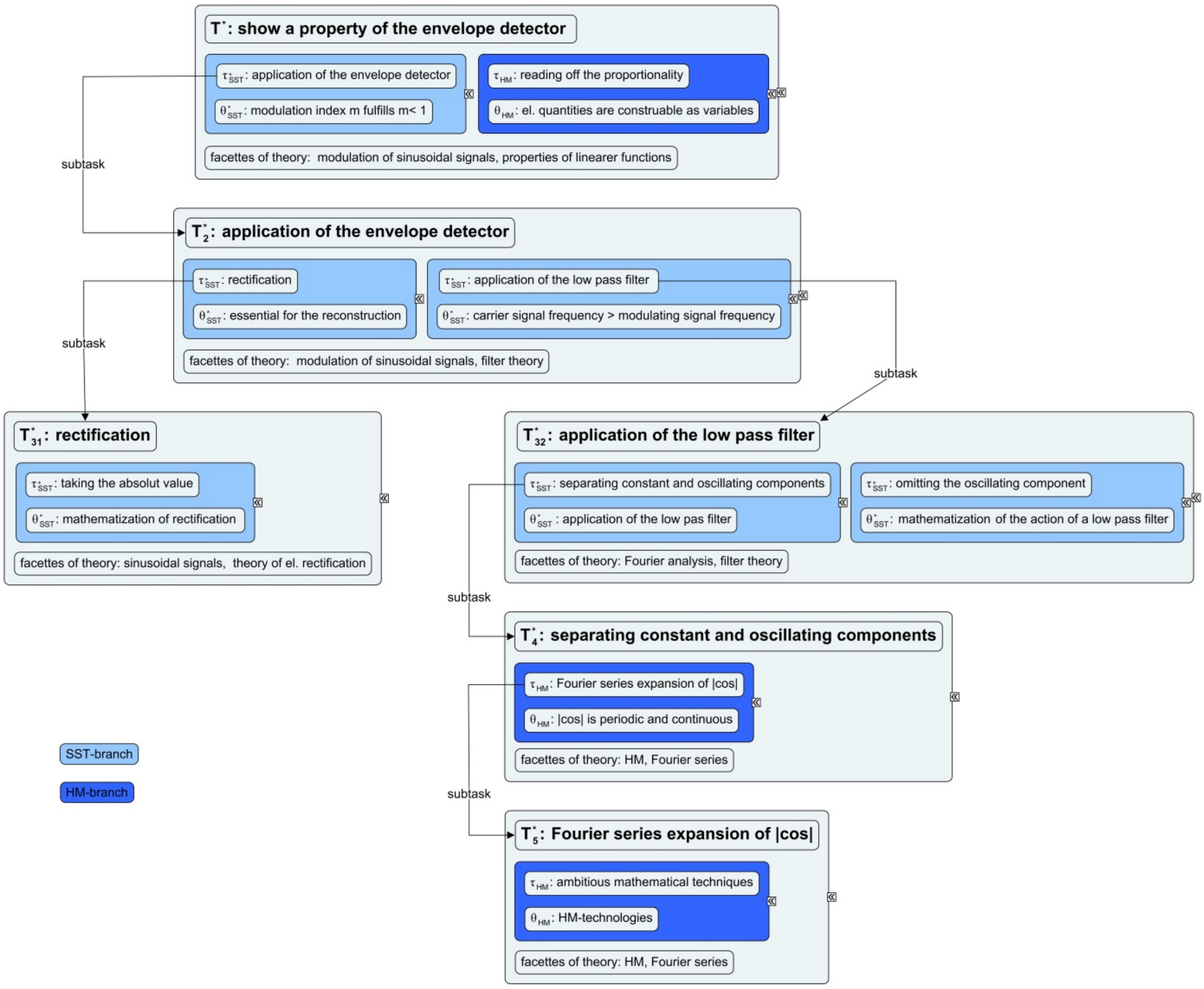

Figure 1: Graphical visualization of the analysis

Discussion and Outlook

The analysis in the preceding section shows, that the extended praxeological model is in principle capable of discriminating different discourses. It is noticeable, that the full Fourier series expansion of the signal is not necessary to solve the task. It would be sufficient to calculate the first Fourier coefficient, because the subsequent application of the low pass filter would suppress further summands of the series. In this way, by keeping electrotechnical aspects of the problem in mind, the solution would be much shorter and much more effective. For example, the ambitious HM-techniques, involving manipulation of infinite sums, wouldn't be necessary any more. Calculating the first Fourier coefficient involves only simple integral techniques.

A first hypothesis resulting from the analysis is that by keeping electrotechnical reasoning patterns and justifications in mind, at least some exercises could be solved more effectively. So, being able to recognize the different mathematical discourses in SST-courses would enable students to determine effective solution steps.

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Appendix

In amplitude modulation (AM) the amplitude of a high frequency signal ("carrier signal") is modulated to carry a low frequency signal ("modulating signal") for transmission. An AM-signal can be represented as:

$$x(t) = A(t)\cos(2\,\pi\, f_0 t) = A[1 + m s_1(t)]\cos(2\,\pi\, f_0 t)$$

with modulating signal $s_1(t) = \cos(\Omega\, t)$, $\Omega \ll 2\pi f_0$ and modulation index m. Under certain conditions (AM-signal with $0 < m < 1$), an envelope detector, a simple circuit consisting of a rectifier and a low pass filter, can be used to reconstruct the modulating signal at the receiver.

Sample solution The envelope detector gives the signal $|x(t)| = |A(t)\cos(2\pi f_0 t)|$. Because of $0 < m < 1$ thus $A(t) > 0$ we have $|x(t)| = A(t)|\cos(2\pi f_0 t)|$. The periodic function $|\cos(2\pi f_0 t)|$ can be expanded in a Fourier series. We consider the complex Fourier series, which is used in the lecture, and set

$$s(t) = |\cos(2\pi f_0 t)| \overset{!}{=} \sum_{n=-\infty}^{\infty} S_n e^{jn2\pi Ft}$$

With period T of $s(t)$ with $T = \frac{1}{2f_0}$, thus $F = \frac{1}{T} = 2f_0$. The coefficients of the Fourier series are defined by

$$S_n = \frac{1}{T}\int_{-T/2}^{T/2} s(t)e^{-jn2\pi Ft}\,\mathrm{d}\,t = \frac{1}{T}\int_{-T/2}^{T/2} s(t)[\cos(2\pi nFt) + j\sin(2\pi nFt)]\,\mathrm{d}\,t$$

$s(t)$ is an even function, but $\sin(2\pi nFt)$ is uneven, thus

$$S_n = \frac{1}{T}\int_{-T/2}^{T/2} s(t)\cos(2\pi nFt)\,\mathrm{d}\,t$$

integrand is even

$$= \frac{2}{T}\int_{0}^{T/2} s(t)\cos(2\pi nFt)\,\mathrm{d}\,t$$

$$= \frac{2}{T}\int_{0}^{T/2} |\cos(2\pi f_0 t)|\cos(2\pi nFt)\,\mathrm{d}\,t$$

$$= \frac{2}{T}\int_{0}^{T/2} \cos(2\pi f_0 t)\cos(2\pi nFt)\,\mathrm{d}\,t$$

addition theorem, $F = 2f_0$

$$= \frac{1}{T}\int_{0}^{T/2} [\cos(2\pi f_0(2n+1)t) + \cos(2\pi f_0(2n-1)t)]\,\mathrm{d}\,t$$

$$= \frac{1}{2\pi T f_0}\left[\frac{1}{2n+1}\sin(2\pi f_0(2n+1)t)\Big|_0^{T/2} + \frac{1}{2n-1}\sin(2\pi f_0(2n-1)t)\Big|_0^{T/2}\right]$$

$2f_0 T = 1$

$$= \frac{1}{\pi}\left[\frac{1}{2n+1}\sin\left(\frac{\pi(2n+1)}{2}\right) + \frac{1}{2n-1}\sin\left(\frac{\pi(2n-1)}{2}\right)\right]$$

Due to $\sin\left(\frac{\pi(2n+1)}{2}\right) = (-1)^n$ and $\sin\left(\frac{\pi(2n-1)}{2}\right) = (-1)^{n+1}$, $n \in \mathbb{Z}$ we get

$$\begin{aligned} S_n &= \frac{1}{\pi}\left[\frac{(-1)^n}{2n+1} + \frac{(-1)^{n+1}}{2n-1}\right] \\ &= \frac{(-1)^n}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right] \\ &= \frac{(-1)^n}{\pi(2n+1)(2n-1)}[(2n-1)-(2n+1)] \\ &= \frac{(-1)^n}{\pi(4n^2-1)}(-2) \\ &= \frac{2}{\pi}\frac{(-1)^{n+1}}{4n^2-1} \\ &= S_{-n} \end{aligned}$$

And therefore

$$s(t) = \sum_{n=-\infty}^{\infty} S_n e^{jn2\pi Ft}$$

$S_n = S_{n-1}$

$$\begin{aligned} &= S_0 + \sum_{n=1}^{\infty} S_n \left(e^{jn2\pi Ft} + e^{-jn2\pi Ft}\right) \\ &= \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{4n^2-1}\cos(4\pi n f_0 t) \\ &= \frac{2}{\pi}\left[1 + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{4n^2-1}\cos(4\pi n f_0 t)\right] \\ &= \frac{2}{\pi}\left[1 + \frac{2}{3}\cos(4\pi f_0 t) - \frac{2}{15}\cos(8\pi f_0 t) \pm \cdots\right] \end{aligned}$$

If the low pass filter is applied to the signal

$$|x(t)| = A(t)s(t) = A(t)\frac{2}{\pi}\left[1 + \frac{2}{3}\cos(4\pi f_0 t) - \frac{2}{15}\cos(8\pi f_0 t) \pm \cdots\right]$$

we get on the output side

$$y(t) = \frac{2}{\pi}A(t)$$

a signal proportional to $A(t)$.

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The role of mathematics in engineering education

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While problem solving has been researched in depth, little is known about how engineering students' develop domain-specific problem solving competences during their qualification. In our study, we opted for investigating the interplay between mathematics and physics in terms of an outer and inner structure of students' problem solving behavior. In our study, 21 engineering students at the beginning of their studies participated. Data was collected by paper-and-pencil tests focusing on mathematical and physical content knowledge as well as by videotaping group settings, immersing students in problem solving. We found different performance levels that can be characterized by students' problem solving skills and how their mathematical and physical knowledge interact.

Introduction

Mathematics is an important subject in engineering education. The first terms of study are characterized by a high usage of mathematics, be it in the mathematics lectures themselves, in physics or further engineering lectures. Besides continuously improving their declarative and procedural knowledge in the respective fields, it is important for students to develop their problem solving competencies as this is one of eight competencies that engineering students need to learn according to the SEFI (2013) framework. The formation of these competencies is, however, often hampered by an asynchronicity of mathematical and engineering education. Overarching aim of the project KoM@ING is to measure mathematical competencies of engineering students and the relations between the different lectures by combining a quantitative and a qualitative approach. In the quantitative project (project partners from IPN Kiel and University Stuttgart), IRT-based measures for higher mathematics and technical mechanics are developed to capture students' development in their first year of study. Thus, individual competencies are measured reliably and validly, but no insight is provided into the concrete problem solving processes of students (Neumann, et al. 2015; Behrendt, et al. 2015). This presentation elaborates on the work of the qualitative project that scrutinizes these processes. That is, item difficulties as revealed by the IRT-measures are discussed in view of the problem solving phases involved, the employment of different heuristics, and the epistemic games that mediate between the world of physics and mathematics.

Theoretical framework

In recent years, researchers have developed many different approaches to conceptualize problem solving in the domains of mathematics and physics (cf. Schoenfeld, 1985; Heinze, 1997; Rott, 2013; Redish, 2005). These different perspectives on problem solving can be classified by distinguishing an *outer structure* in terms of timing and organizing cognitive processes and an *inner structure* considering heuristics and beliefs (cf. Philipp, 2012). In this

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paper, we will explore the outer structure of students' problem solving by drawing on Polya's (1945) phase model of problem solving processes and by considering *Epistemic Games* (Tuminaro & Redish, 2007) when it comes to problem solving in physics contexts. In addition, we focus on the inner structure of students' problem solving by referring to *heuristic tools, heuristics strategies* and *heuristics principles* (cf. Bruder & Collet, 2011).

Modeling the process of (students') problem solving from a more general perspective, the seminal work by Polya (1945) provides a promising approach by distinguishing four phases: *understanding the problem; devising a plan, carrying out the plan, looking back.* Polya's influential model has been the basis for much of the research that has been conducted in the field of mathematical problem solving processes. Schoenfeld (1985) for example extended Polya's model by an *exploration* phase, connecting the initial phase of understanding a problem with devising a plan. Likewise, Chinnappan and Lawson (1996) stress the importance of the first two phases: "[...] the planning process forced the solver to make optimum use of information that was identified and information that was generated" (p. 13).

Concerning the interplay between mathematics and physics, Redish (2005) stresses that in physical contexts mathematics is used to "describe, learn about, and understand physical systems" (p. 6). Hence, to solve problems in physics in general one conducts the four steps of mapping, processing, interpreting and evaluating that moderate between the two different worlds of mathematics and physics (figure 1):

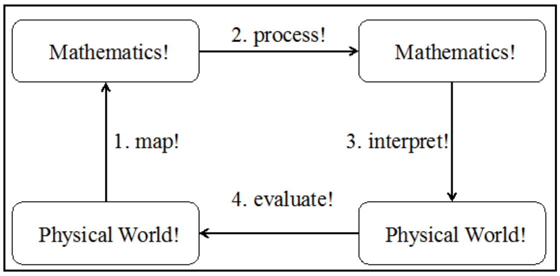

Figure 1: A model for the use of mathematics in physics (Redish 2005).

In order to analyze students' problem solving in physics in detail, Tuminaro and Redish (2007) draw on the above-mentioned ideas and suggest a framework called *Epistemic Games* which describes the outer structure of mathematical problem solving processes in physics contexts: "An epistemic game has a goal, moves (allowed activities), and an end state (a way of knowing when the game has been won)" (Redish 2005, p. 8). Hence, an *Epistemic Game* can be described by three components: *entry*, *ending* conditions and allowed *moves*. The entry and ending conditions are determined by students' expectations about and experience with physical problem solving. That is, as students are able to categorize physical problems very quickly, this influences their choice of playing a specific game. The allowed moves are the "steps and procedures that occur in the game" (Tuminaro and Redish 2007, p. 12). In particular the authors identify six different epistemic games involved in students' problem solving processes: *Mapping Meaning to Mathematics*, *Mapping Mathematics to Meaning*, *Pictorial Analysis*, *Physical Mechanism Game*, *Recursive Plug-and-Chug* and *Transliteration to Mathematics*.

In addition to the outer structure of the solution process, the methods, heuristics and strategies used during the problem solving process are crucial for a successful performance as well. Chinnappan and Lawson (1996) describe the cognitive actions involved in the problem solving process in detail by distinguishing task-specific actions like "the 'invert and multiply' procedure in division of fractions problem" and domain-specific and domain-related actions like "draw a diagram" (p. 2). Bruder and Collet (2011) pursue this approach, too, and explain problem solving by means of heuristic tools, heuristic strategies and heuristic principles.

Based on the theoretical framework described above in our study, we explore engineering students problem solving behavior in the first term of their studies and pay attention to the inner and out structure of the processes involved. In particular we pursue the following research questions:

- What general differences can be detected within different performance groups in terms of the outer structure and inner structure of the problem solving processes at the beginning of the first semester?
- What differences in terms of the inner and outer structure of problem solving can be detected within a group of low performers and a group of high performers after the first term of engineering studies?

Methodology

In our study N = 21 students from one university in Germany in their first year of engineering studies participated. Among them were 6 female and 15 male students covering an age range from 18 to 21 (M_{age} = 19.29 years; SD_{age} = 0.96 years). While all participants attended university courses considering fundamental knowledge in mathematics and physics, part of the students also attend courses in mechanical engineering (n = 15) and electrical engineering (n = 3); the remaining students indicated another or no discipline of their studies.

Engineering students' problem solving competence was measured two times in the course of the first term of their studies (cf. table 1). First, participants in our study worked alone on IRT-scaled tests for mathematical content knowledge (MathCK) and physical content knowledge (PhyCK), delivered from the quantitative sub-project of KoM@ING. In addition, students filled in the Force Concept Inventory which is an instrument for capturing physics-related beliefs about the force concept. Second, students worked in groups (N_{Group} = 9) of two or three on a set of 16 items of the MathCK-test and on a set of 13 items of the PhyCK-test with varying difficulty. The group work setting was chosen to initiate discussion among students and thus to allow for displaying their thinking processes (cf. Thinking Aloud-method, e.g. Ericsson and Simon 1984). At the second point of measurement, the students worked within the same groups.

Results and discussion

Based on students' content knowledge in mathematics and physics, a classification of three different groups could be applied to our initial nine groups: (1) high performers, (2) medium performers, and (3) low performers. Taken together, our findings show that students with a different level of professional knowledge in mathematics and physics use different situation-specific skills. In terms of the outer structure of the problem solving process, perceiving and interpreting a given problem is highly influenced by students' dispositions. The low performers are not able to take advantage of the mathematics and physics lectures of the first term as do the medium and high performers. Although they possess more mathematical and physical resources at the end of the first term, they are still not able to apply their knowledge and to connect different facets to solve the various problems. High performers are successful because they consider more consciously than the other students all problem solving phases in more complex tasks. Their higher content knowledge in mathematics and physics enables them to apply mathematical procedures and physical laws for tasks that require that. These procedures and laws are not spontaneously available for the medium and low performers. In particular, the situation-specific skills reveal the interplay of mathematics and physics content knowledge as decisive. Remarkably, we found that the use of "mathematics-dominated" Epistemic Games like Mapping Meaning to Mathematics could only be observed in the solution processes of the high performers and the medium performers. In the group of low performers no group used these kinds of Epistemic Games, even though it would have been convenient to do so. When taking a closer look at the game *Mapping Meaning to Mathematics* in comparison to the game *Mapping Mathematics to Meaning*, our findings generically show that the group of high performers and the group of medium performers approached this specific Epistemic Game differently: While the group of high performers more frequently engaged in the game *Mapping Meaning to Mathematics,* the group of medium performers more often used *Mapping Mathematics to Meaning*, implementing the contrary thinking process. The different use of Epistemic Games point to advantages that high performing groups have due to elaborated content knowledge, enabling them to approach tasks on a meta-level.

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Mathematics in economics study programmes in Germany: structures and challenges

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At German universities there are two different approaches to mathematics education of students in business administration and economics (BE) study programmes: on the one hand, traditionally, mathematics modules are offered by mathematical departments (external mathematics education (exME); mathematics as a service subject). On the other hand, in many cases, the mathematics education of BE students is organized by BE departments (internal mathematics education (inME)). Thus, interesting questions arise: How do the approaches differ structurally? Which approach is better? What conditions are crucial? Which approach is more appropriate to address current and future challenges? This article attempts to give first answers for German universities based on data from 66 mathematics modules.

Introduction

Today, mathematics plays an increasing role in almost all scientific disciplines and, in particular, to business administration and economics (BE). Two developments have contributed to this fact. First, the application of mathematical techniques in economic theory has increased significantly since the 1940s. Secondly, empirical studies and simulation studies have gained importance in recent decades due to the availability of technical resources (computer hardware and software) and data. A look at textbooks on mathematics for economics, micro- and macro-economics and econometrics shows the variety of mathematical techniques applied in BE study programmes. Against this background, it is no surprise that mathematics modules are anchored in all BE study programmes, where the curriculum exceeds the level of secondary-school mathematics. However, the corresponding modules differ considerably in terms of scope and content. Obviously, there are also large differences in terms of the professional and academic background of the mathematics educators.

In general, two different approaches are implemented in BE study programmes with respect to responsibility for mathematics education. One approach is where economics departments provide the mathematics modules using mathematics educators from the BE departments (internal mathematics education (inME)). In the other approach, the mathematics education is provided by mathematical departments (external mathematics education (exME)).

Against this background, important questions arise:

- Why have these two approaches emerged? In some disciplines, exME does not prevail (e.g., psychology), in other disciplines, imME does not exist.
- What are the advantages and disadvantages of inME and exME in the field of BE?
- What is the more appropriate concept regarding current and future challenges?

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In this paper only some of these questions can be answered. The following empirical analysis is based on structural data from 66 mathematic modules integrated in 98 BE study programmes (business administration (BA), economics (ECO), and business administration and economics (BA&ECO)) at German public universities. The focus is on the structural differences between inME and exME. Moreover, in an eclectic manner, current and future challenges are specified that are likely to affect the adequacy of inME and exME.

Data

Public German universities were considered if they offered one of the three study programmes: BA, ECO or BA&ECO. Thus, 62 universities were considered, in which 66 modules were offered for a total of 89 study programmes. The data have been taken from module descriptions and examination regulations. In addition, the web pages of the relevant departments, institutions and faculty members were used for the database. The data were collected for the academic year 2014/15.

Structures: An Overview

The 66 modules were offered for a total of 89 study programmes (BA n = 35; ECO n = 28; BA&ECO n = 26). With four exceptions, one module was offered at each university, even if several study programmes were offered. In about half the cases (n = 32), the module was divided into two sub-modules. Thus, a total of 34+2×32=98 sub-modules were considered. Note: If a module was not divided then the module is also called a sub-module by definition.

When a module was divided, the two sub-modules were usually offered by the same mathematics educator. In nearly all cases, the module was scheduled for the first semester or the first and second semester if the module was divided. On average, 8.5 ECTS credit points were awarded for mathematics. The average number of contact hours was 6.6 per week. Subjects of mathematics modules were secondary-school mathematics as well as basic topics of analysis and linear algebra. Usually, financial mathematics was integrated. In some cases, a separate module on financial mathematics was offered. However, these separate courses are not part of this analysis.

Nearly half of the modules (n = 30; q = 45%) were offered by mathematical departments (exME), slightly more than half (n = 35; q = 53%) by BE departments (inME). In one case, a mixed department (BE and Mathematics) was responsible for the mathematics module.

Of the 98 sub-modules, 51 were offered by mathematics educators who had attained the academic title of professor. In five cases, mathematics educators did not have a doctoral title (e.g., Dr, PhD). However, the academic title of a mathematics educator alone does not allow conclusions to be drawn about the professional status of the mathematics educators. Since information regarding the professional background of the mathematics educators is often not publicly available or classification problems are present, no further information exists. For that reason, information about the share of mathematics educators who held a permanent position was not available using the data sources mentioned above.

Moreover, the collection of data on academic careers is difficult because CVs are often not publicly available. In 53 of the 98 cases it could not be determined which first academic degree (e.g., master's degree) the mathematics educators had obtained. In 29 of the remaining 45 cases (q = 64%) the mathematics educator held a degree in mathematics. Five mathematics educators (q = 11%) held a degree in economics. Academic degrees in physics (n = 3), mathematical economics (n = 3), information management (n = 2) and statistics (n = 1) played a minor role. Note: each mathematics educator was counted twice if she/he offered both sub-modules of a module.

Of the 98 mathematics educators of sub-modules, 93 held a second academic degree (PhD or a similar). In 48 cases, no information existed about the doctoral thesis field. In 23 of the remaining 45 cases the mathematics educators held a PhD or a similar degree in mathematics (q = 51%) and 12 in economics (q = 27%). In up to three cases the mathematics educators had gained a PhD in business administration (n = 3), geology (n = 1), engineering (n = 1), physics (n = 2), statistics (n = 2) or information management (n = 1).

At a third academic level (second doctoral degree; e.g., habilitation and comparable qualifications), data availability was even more limited. It was assumed that 54 of the 98 mathematics educators offering a sub-module had achieved a second doctoral degree. In 22 cases, no further information was found. For the remaining 32 cases, a second doctoral degree in mathematics was assumed that 17 cases (q = 53%). A second doctoral degree in economics was found in nine cases (q = 28%). In the remaining cases a second doctoral degree was held in business administration (n = 3), geology (n = 1) or statistics (n = 2).

Eighteen of the 98 sub-modules (q = 18%) were offered by female mathematics educators. All five mathematics educators without a doctoral degree were female. Seven of the 42 mathematics educators with a PhD but no academic title of professor, were female (q = 17%). In addition, only six of the 45 (q = 11%) female mathematics educators held an academic title of professor.

Structures: Internal vs. External Mathematics Education

Disaggregated results are presented here. Two groups of modules can be distinguished: those offered by mathematics departments (exME, n = 30) and those by economics departments (inME, n = 35). One module was offered by a mixed faculty. Table 1 on the next page shows some initial results.

Major differences were found between external and internal education. Briefly summarized, for external education (compared to internal education) the extent of education was higher, mathematics educators had a better professional position, lectures had a stronger academic background in mathematics, curricula were more homogeneous and textbook selection was broader.

Valuation of Structures

The structures of mathematics education are very heterogeneous. The reasons are manifold. They cannot and should not be discussed at this point. Only one remark: assuming that existing structures are a result of strategic decisions and taking traditional considerations of the theory of strategic management into account, two perspectives exist: One, following the resource-based approach, the resources available to universities are essential; second, the Industrial Organization approach highlights the importance of 'market conditions'.

The structures of mathematics education in economics affect (indirectly) the success of students in BE study programmes. Studies that examine this relationship theoretically or empirically in detail do not exist. For that reason, further research is necessary to evaluate both models and provide advice to decision makers at universities. However, an evaluation of these models will be very demanding. Two aspects should be noted. First, the choice of a success variable is disputed. Second, a model of mathematics education that is adequate today may be inadequate in future because there are many current and future challenges that may change the models' rating.

Table 1: Structures in exME and inME

Variable	**Units**	**exME**	**inME**
Credit points (average)	Modules	9.1	7.7
Contact hours (average)	Modules	7.2	5.8
Share of divided modules	Modules	.60	.37
Share of modules offered by different lecturers	Divided Modules	.06	.31
Share of lecturers with academic title of professor	Sub-modules	.79	.43
Share of lecturers with first academic degree (e.g., master's) in:	Sub-modules		
... mathematics		.77	.50
... statistics		.07	-
... mathematics in economics		.07	-
... economics		-	.22
... business administration		-	.11
... information management		-	.06
... physics		.07	.11
Share of lecturers with second academic degree (e.g., PhD) in:	Sub-modules		
... mathematics		.71	.18
... statistics		.14	-
... mathematics in economics		-	-
... economics		-	.53
... business administration		-	.12
... information management		-	.06
... physics		-	.12
... geology		.07	-
... engineering		.07	-
Curricula:	Modules		
... 'degree of heterogeneity'		(Low)	(High)
... further modules with mathematical content		(No)	(Yes)
Textbooks:	Modules		
... 'degree of heterogeneity'		(High)	(Low/High)
... use of Sydsaeter et al.		(Low)	(High)

Challenges

In this section, some hypotheses on current and future developments are mentioned in an eclectic form. The question is which of the two models of mathematics education will resolve the challenges in a satisfactory manner.

- Mathematics in economics increasingly loses the character of a preparatory discipline.
- Due to the implementation of the Bologna decisions, a process of modularization has occurred. As a rule, modules of a study programme have become self-contained and non-consecutive.
- In almost all areas of economics, a process of mathematization occurs. In BE this is true for research as well as professional practice.
- The importance of statistical and numerical methods and, hence, the use of appropriate software (e.g., R, Stata, Mathematica) increases.
- In parts of economics, the current use of mathematics is being critically discussed. Representatives of 'plural economics' are demanding the use of more appropriate mathematical methods.
- The heterogeneity of students increases with regard to numerous factors. This applies to socio-economic, biographical, educational, psychological and motivational factors.
- Education policy decisions in secondary school education have led to changes in mathematical skills in Germany (for example, the transition from G9 to G8 and the introduction of new education standards).
- After education policy decisions in higher education in several German states, a study programme at a university can be taken without a general qualification for university entrance (in Germany: Abitur). Usually, these students have below-average mathematics skills that make additional training in basic mathematics necessary.
- In some German states, premiums for universities are discussed. For each successful study completion, the university should receive a premium. This could mean the importance of mathematics modules will be reduced because mathematics is considered too demanding.
- The scientific competition of economics departments increases in research. There is evidence that mathematically oriented departments have an advantage. This implies a higher level of mathematics in BE study programmes.
- The competition of economics departments to students is also increasing. Depending on the strategic decisions, the importance of mathematics in economic study programmes may increase or decrease.

These developments concern mathematics in BE in general. However, it remains to be analyzed which model of mathematics education may withstand the challenges.

Conclusion

This paper has shown that the structures of mathematics education in BE study programmes at German universities differ significantly. In particular, differences between exME and inME are evident. Furthermore, current and future challenges were outlined. Until now it was not clear under which conditions exME (with respect to inME) was more appropriate. This should be clarified in further research. Clearly, the result will depend on the choice of educational and economic target variables and the relevant educational and economic constraints.

Application-oriented tasks for first-year engineering students

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Our poster will present a concept for the construction of special application-oriented tasks for first-year engineering students (especially for mechanical engineering) and a study about these tasks. The main goals of these tasks are to create a connection between mathematics and physics / engineering and the students' interests in order to increase motivation, understanding of mathematical contents and the relevance-rating of mathematics for engineers. The study indicates the students' attitudes about our tasks and gives hints to optimize the concept and the tasks.

About the project

The project team "Ing-Math" of the competence center of university didactic of mathematics (khdm) is currently realizing three projects. The project we would like to present is called "Mathematik für Maschinenbauer: Integration des Modellierens in ingenieurwissenschaftlichen Zusammenhängen" (German for "Mathematics for mechanical engineers: Integration of modelling in engineering teaching"). The project group works under the direction of Rolf Biehler and Gudrun Oevel, see also OEVEL et al. (2014).

In our project we develop and implement interventions for the lectures of mathematics for mechanical engineering and analyze them in regard to their acceptance and effects. The planned interventions are especially:

- Emphasizing the application area of mathematics in engineering: Preparing the students for simulating, modelling and interpreting of problems and their solutions
- Visualization and integration of mathematics through engineering applications
- Reorganization of the learning content in time and order, so the required mathematics are taught parallel to the engineering lectures
- Reorganization of the learning content regarding their relevance
- Carrying out empirical studies about the effects of our interventions and about the attitudes and competencies of the students

Our poster will mainly present our work on the first, the second and the last point. The developed tasks and their conception as well as their enhancement, evaluation and the background theories will be a main part of the dissertation of the first author.

Our conception for developing application-oriented tasks

Many first-year mechanical engineering students don't know and cannot see the connections between mathematics and their field of study. In many cases demotivation and disin-

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terest on mathematics are consequences – although math is an important subject for engineers.

The goal of the project is to develop and evaluate tasks which fit in usual mathematic lectures for mechanical engineering regarding to the topics and the requirements. But what does “fit in” mean in this case? We will present the criteria, which he have developed. These criteria are on the one hand anchored in the discussion of the mathematic didactic about classification of application-oriented tasks (MAASS, 2010) and on the other hand they consider the specific conditions of mathematic lectures for engineers. With regard to the workload the tasks should be equivalent to one out of usually four exercises in a week. Thereby we distinguish our project from other ones which are more time-consuming and more complex (like ROOCH et al. (2014) or ALPERS (2001)). In comparison our idea is less extensive, but less expensive and can be implemented in nearly every usual mathematic lecture for engineers.

The development of the tasks is based on the conception which is currently developed by the first named author in the context of his dissertation project.

Figure 1 shows the most important aspects of our conception “good application-oriented tasks in mathematics for mechanical engineers” in short. At the poster-session we will gladly explain every aspect, the terms as well as their relevance and practicability.

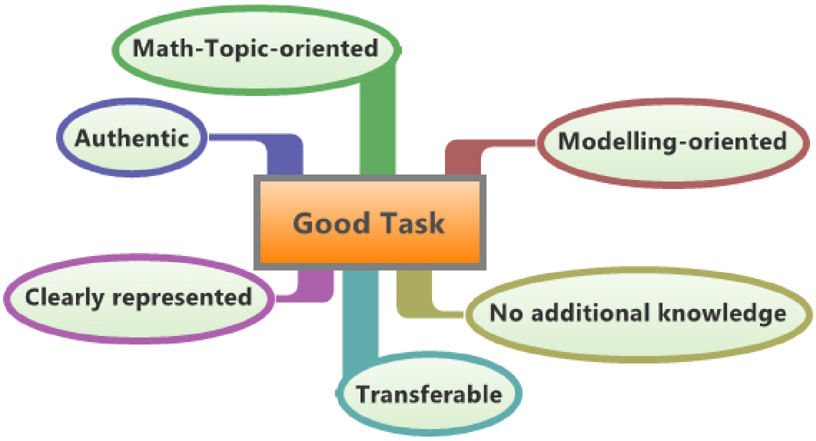

Fig. 1: The conception in short

Most of our tasks, like “Half pipe”, “Laser beam” and “Stressing a structural element”, are published in Wolf & Biehler (2014, German). Some tasks are already translated in English and can be enquired via wolf@khdm.de.

Testing and evaluating the tasks

In 2013, in order to evaluate the tasks, we observed and filmed some students while they were solving one of our tasks. The data has been analyzed within the scope of a master thesis and the results indicated that the development of such application-oriented tasks is promising. In 2013/14 we carried out a comparative study between mechanical engineering (treatment group) and economical engineering (control group) students who were visiting the same mathematical lecture. While the treatment group additionally worked on our application-oriented tasks every two weeks, the control group just solved pure mathematical tasks as usual. Through surveys at the beginning, during and the end of the semester we got a comprehensive view of the students' attitudes towards our tasks and about the effects of the intervention. In 2014/15 we carried out a similar study, but this time over two semesters and the treatment group was randomly chosen (no distinction between the different fields of study). In both studies we were able to prove that our tasks have had positive effects on the students' appreciation for the relevance of the mathematical topics. Especially the lack of such tasks causes a decrease in motivation. Round about 62% of the control group and only 20% of the treatment group would like to switch to the other group. Furthermore our surveys showed that over 80% of the treatment group would prefer an application-oriented tasks to a pure mathematical task with the same topic.

In the course of our studies we have seen that the students want a closer combination of the mathematical and the technical / physical lectures of their field of study. Due to that fact we initialized a cooperation between the lecturers, which lead to regular meetings, new research possibilities and a matching of the content of teaching.

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4. TERTIARY LEVEL TEACHING (ANALYSES, SUPPORT AND INNOVATIONS)

University students' eye movements on text and picture when reading mathematical proofs

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Theories of multimedia learning suggest that the combination of text and picture can support learning. It is still an open question whether this combination is also beneficial for learning mathematics and particularly for understanding mathematical proofs. It is also unclear whether learners actually look at pictures when reading proofs. We analyzed eye movements of 19 university students with low and high prior knowledge while they were reading three proofs. We found that the students looked at the pictures, but also that they fixated longer on the text than the pictures, and that they alternated between text and pictures. We also found a weak tendency that less experienced students focused longer on the pictures and switched back to the pictures more often than more experienced students.

Theoretical Background and Research Question

The use of pictures that accompany text is common practice in university teaching. Cognitive psychological theories on multimedia learning (e.g. Mayer, 2001; Schnotz & Bannert, 2003) give explanations why it can be reasonable to combine text and pictures to facilitate learning. In their integrated model of text and picture comprehension, Schnotz and Bannert (2003) differentiate between descriptive (e.g. text, formulas, mathematical expressions) and depictive (e.g. drawings, diagrams, maps) representations. When reading a text, readers first create an internal representation of the text surface (descriptive), then a propositional representation of the text (descriptive), and finally a mental model of what they have read (depictive). To comprehend pictures, readers first create a visual image of the picture (depictive), then a mental model (depictive), and finally a propositional representation of what they have seen (descriptive). That means that external descriptive representations as well as external depictive representations lead to both internal descriptive and internal depictive representations (Schnotz & Bannert, 2003). Therefore it seems to be reasonable to integrate information from descriptive and depictive representations in teaching materials to give the readers the opportunity to build up a rich mental model respectively propositional representation of the content. Accordingly, for the reader, alternating between text and pictures seems to be a good reading strategy to make adequate mental representations of the presented materials.

In fact, the combination of text and pictures seems to be effective for the learning process, especially for learners with low prior knowledge (Schnotz & Bannert, 2003). Although a large number of studies have focused on multimedia learning, there is only little empirical research on multimedia learning in the domain of mathematics (Atkinson, 2005). In a recent study by Beitlich, Obersteiner, and Reiss (2015) the authors analyzed how secondary school students make use of different representation formats (text, mathematical symbols, and

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pictures) in heuristic worked examples. The authors found that the students spent most of their reading time on looking at the pictures, followed by symbols, and text. They also found that the students alternated between different representation formats relatively often which might be an indicator that they were trying to integrate information from different representation formats.

At university, mathematics is typically taught in a definition – theorem – proof structure. That means university students' central activity is reading and constructing mathematical proofs (Mejia-Ramos & Inglis, 2009). In the last years, an increasing interest in studying the reading of mathematical proofs evolved. In a study by Inglis and Alcock (2012) undergraduates and mathematicians had to read proofs in order to validate them while their eye movements were recorded. Besides other results, the authors found that the students spent significantly more time on the formula parts of the proofs than the mathematicians. Furthermore the mathematicians shifted their attention back and forth between consecutive lines of the proof significantly more often than the students.

Even though mathematical proofs play a very important role in their studies, many students struggle with mathematical proofs. One way to support students' learning is combining textually presented mathematical proofs with pictures. Even if the combination of text and pictures within proofs is often found in textbooks and lectures (Stylianou, Blanton, & Rotou, 2015), it is still an open question whether such an approach can in fact enhance students' proof comprehension. More specifically, it is to date unclear whether students actually make use of such pictures at all when reading a mathematical proof, and, if so, whether the beneficial effects of multimedia learning found in other disciplines also hold for learning of mathematics. There are two reasons why studying both questions is particularly relevant in the domain of mathematics. First, it is not clear that learners take into account pictures at all when the textually presented mathematical problem contains all relevant information. For example, Dewolf, Van Dooren, Hermens, and Verschaffel (2015) found that university students scarcely looked at pictures presented together with word problems. Second, mathematical objects are abstract and although using visual representations of these objects might help understanding relationships between them, arguments based on visualizations are not accepted in a mathematical proof.

A suitable method to address the above formulated questions is eye tracking. More and more studies on multimedia learning have successfully used this method, because it allows conclusions about which pieces of information people take into account during the learning process (Van Gog & Scheiter, 2010). Eye movements consist mostly of fixations and saccades. During a fixation, which typically lasts for about 200-300 ms, information is perceived. A saccade is the very fast eye movement between two fixations, during which no information is perceived. There are two assumptions underlying the idea of analyzing eye movements: The immediacy assumption states that processing of information takes place immediately, the eye-mind assumption states that people mainly process the information they are looking at (Just & Carpenter, 1980). Although this strong version of the eye-mind assumption may not hold true in general, it seems to be reasonable to rely on this assumption when people are working on cognitively demanding mathematical problems that contain information in textual and pictorial format.

The study described in the following is an exploratory pilot study to address the question whether university students look at a picture presented with the text of a mathematical proof when they are asked to read the proof. We included students with high and low mathematical skills to also investigate whether the looking on text and picture depends on prior knowledge.

Method

The participants in the study were 19 university students (six female), for whom mathematics lectures were a mandatory part of their studies. Their mean age was 23.3 years (*SD* = 3.3). They were asked to fill out a questionnaire on personal data like their specific subjects (e.g. mathematics or computer sciences) and the number of semesters they had already studied at university. After that, they took part in a short mathematics test. Then the students sat in front of a computer screen, which was connected to an eye tracker. The students were asked to read three mathematical proofs on the screen that were taken out of textbooks on different areas for beginning students of mathematics. The participants were told that they should try to comprehend the proofs. Every item consisted of a theorem, its proof, and a picture that was placed between the theorem and the proof. The picture did not contain any additional information, but represented a part of the information provided in the text. There was no time limit, and the students had to press the space bar to proceed to the next item at their own convenience. After each item an open-ended question about the proof appeared on the screen and the students were asked to write down the main idea of the proof. All responses were given via the computer keyboard.

To answer the research question we analyzed dwell times on the areas *text* and *picture*. For the area *text* we included only the text part of the proof, but not of the theorem. The dwell time is an indicator for how long the eye gaze remains on a specific area. We also analyzed dwell times standardized on the total reading time as overall reading times varied largely between participants. Furthermore we analyzed the order of the gazes on *text* and *picture* by means of sequence charts, which show the order and duration of the gazes on every area of an item for every participant.

We allocated students to low or high prior knowledge groups based on a score calculated from their specific subject, their number of semesters at university, and their result of the mathematics test. There were twelve students in the low prior knowledge group, and seven in the high prior knowledge group.

Results

The dwell times on the area *picture* were above zero for every student in all of the three items. That means that every student looked at least briefly at the picture. To illustrate the dwell times, figure 1 shows a so-called heat map of the gaze durations of one item, summarized over all participants. The colors indicate the least and most fixated areas, where blue stands for the shortest fixation times and red stands for the longest fixation times.

The standardized dwell times showed that the students looked longer on the area *text* than on *picture*. On average the students looked at *picture* for 18% of their total reading time, and at *text* for 71%.

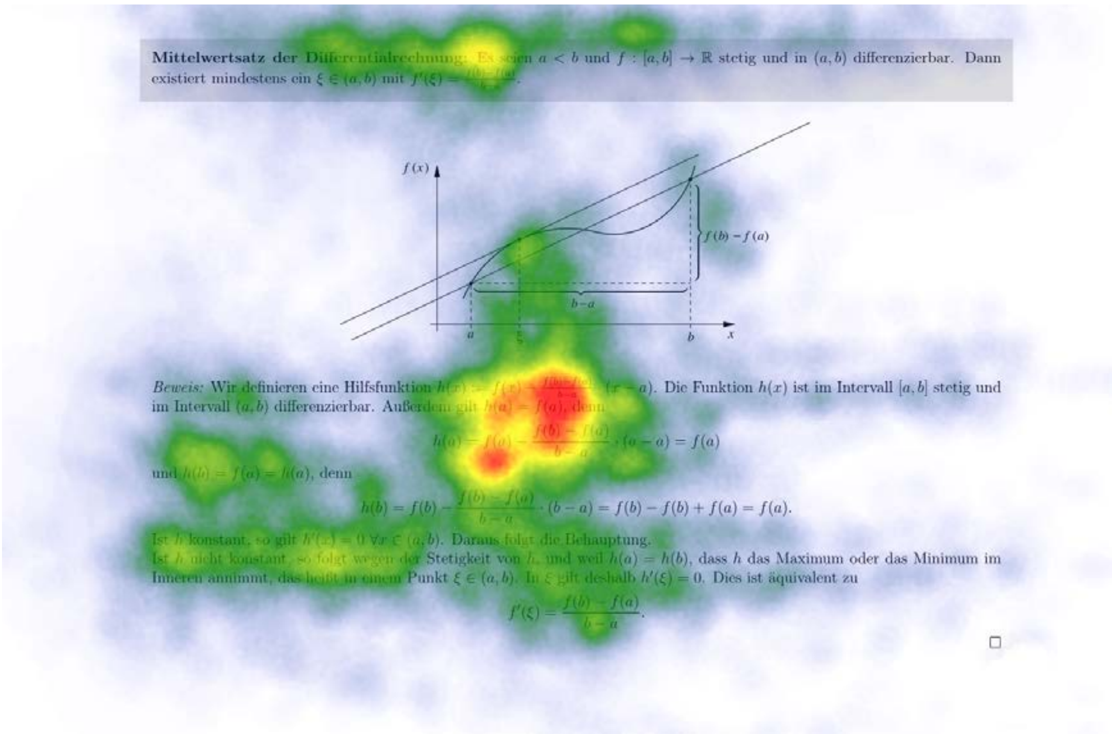

Figure 1: Heat map of the gaze durations of one item, summarized over all participants.

The analysis of the sequence charts showed that nearly all students (except one) switched back and forth between *text* and *picture*. All students started with reading the theorem first. Most of the students continued in the given order (picture, then proof). The others continued with reading parts of the proof before having the first look at the picture. Exemplarily, figure 2 shows the sequence chart of one item. Gazes on the area *text* are colored green, on *picture* orange. The gaps result for example from gazes to other areas like the theorem. The other two sequence charts provide similar information.

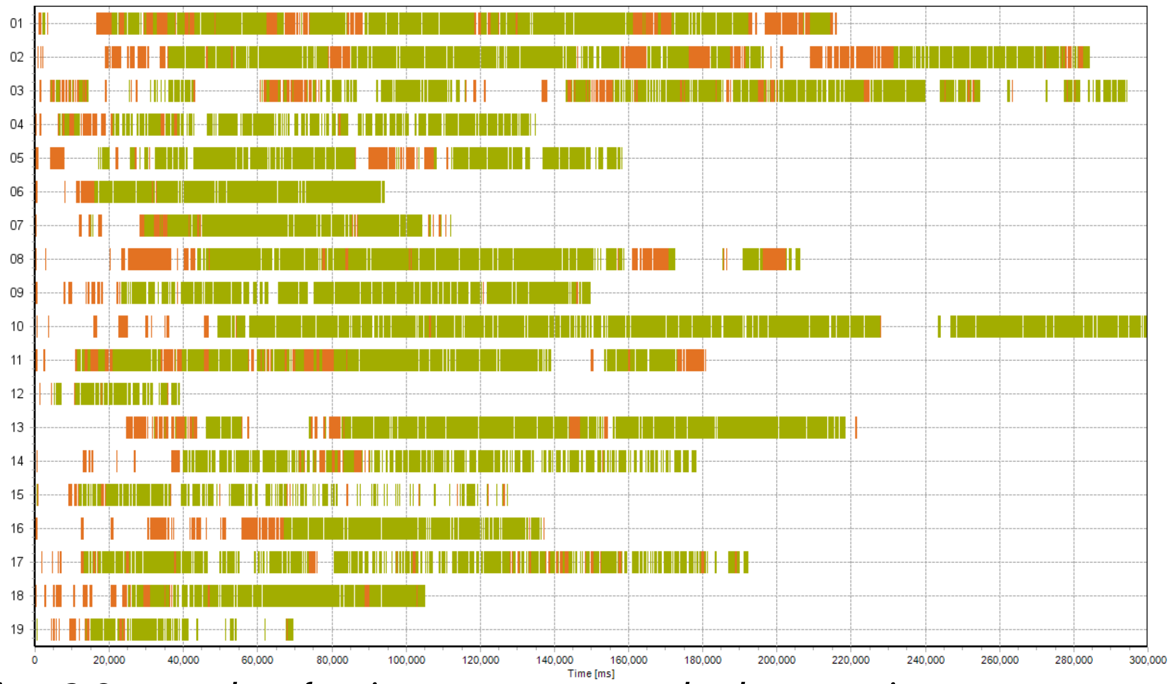

Figure 2: Sequence chart of one item; gazes on text are colored green, on picture orange.

The comparison between the two groups of students showed a very weak tendency in the direction that students with less prior knowledge looked on average longer (19%) on *picture* than the more experienced students (16%), whereas they looked on average shorter (70%) on *text* than the students with higher prior knowledge (72%). This is possibly connected to the finding that students with less prior knowledge switched back more often to *picture* than the students with higher prior knowledge.

Discussion

The university students in our study looked at the pictures that were presented with the text of mathematical proofs during reading the proofs. Fixation times on the text parts were longer than on the pictures, however, the participants switched back and forth between text and picture. These results confirm previous research by Beitlich et al. (2014), whose findings came from a similar study with academics with high expertise in mathematics.

We found a very weak tendency that students with low prior knowledge looked longer on the pictures than students with high prior knowledge (vice versa for text), and that the less experienced students switched back more often to the picture than the more experienced ones. This is only a first, very vague impression. To make some clearer statements about the influence of prior knowledge on multimedia learning of mathematical proofs, more research is needed.

To get better insights into proof comprehension, we are analyzing the additional data we got from the study, namely the questions about the proofs the students had to answer, and the main ideas of the proofs the students had to write down. Results are still outstanding.

The results of the study provide preliminary answers to the questions formulated in the first part of this paper. In particular, the study shows whether and how students look at pictures when reading mathematical proofs. There are important remaining questions, e.g. whether proof comprehension can be enhanced by pictures accompanying the text, and whether the findings of multimedia learning found in other disciplines are valid for mathematics. More studies with a larger sample size and different items are necessary. In the long run, results of this study and other studies of that kind might help to improve the design of university teaching materials.

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Geometry vs Doppelte Diskontinuität?

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The current mathematics curriculum for teacher education at Freie Universität Berlin needs to be overhauled. In close collaboration with the didactics institute, we propose to insert a full semester course on geometric problem solving and theorem proving prior to the usual introductory classes in Analysis and in Linear Algebra. In addition to drafting an effective syllabus, we need to observe very practical side constraints. This experiment could serve as a test case for the transfer from (basic) research into practice.

Status Quo

The status quo in preservice teacher education has developed as a consequence of several ad-hoc reforms. The overall premise has been to design a course of study for mathematics majors and then to select courses for preservice teachers from that catalog.

History

Traditionally, prospective high-school mathematics teachers sat in the same introductory Analysis and Linear Algebra courses as mathematics majors in their freshman year. Several professors teaching these classes, observed a dichotomy between the two cohorts. It was perceived that, essentially, mathematics majors passed while teacher students failed. (Even though recent empirical research says otherwise, e.g., Roloff Henoch, Klusmann, Lüdtke & Trautwein, 2015.) It was decided in the early 2000's to create separate Analysis and Linear Algebra tracks for the teacher students. Due to resource constraints, this double offering could not be maintained for subsequent classes like Algebra & Number Theory or Stochastic.

Current Curriculum

Semester	Main Subject	Minor Subject
1	Linear Algebra 1 (10 CP)	Linear Algebra 1 (10 CP)
2	Linear Algebra 2 (10 CP)	Analysis 1 (10 CP)
3	Analysis 1 (10 CP) Mathematics & Computers (5 CP)	Stochastics (10 CP)
4	Analysis 2 (10 CP) Geometry (10 CP)	Geometry (10 CP)

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5	Algebra & Number Theory (10 CP) Stochastics (10 CP)	Algebra & Number Theory (10 CP)
6	Seminar (5 CP) Bachelor's Thesis (10 CP)	Seminar (5 CP) Mathematics & Computers (5 CP)

The default course of study for preservice teachers with main subject Mathematics respectively with minor subject Mathematics is displayed in the table on the previous page.

These mathematics classes are, of course, accompanied by didactics, pedagogy and further courses.

Issues

There are a number of immediate issues why the current situation must be changed.

- The drop-out rate has remained high, even after the introduction of separate Analysis and Linear Algebra tracks.
- The different tracks are not necessarily taught in a very different way. This is in part due to the fact that these two subjects are highly standardized across (not only German) universities, and it is in part due to external compatibility requirements.
- Still, the two cohorts are separated early, amplifying the effect of stereotype threat as described by Ihme & Möller (2015). The effect is then most visible in the subsequent classes Algebra & Number Theory and Stochastic where the cohorts are mixed again, now with even more severely different backgrounds.
- In exit interviews still most (80-90%) of graduates report to experience the double discontinuity.

Proposed Changes

Goals

The goal of the reorganization of the mathematics courses must be to deal with the above issues, and to allow for a transition from highschool to university mathematics which is meaningful to the preservice teacher. We still want our graduates to develop problem solving, abstraction, and communication skills of university mathematics, but we also want to equip them with a mathematical self-esteem which allows them to take abstract mathematical arguments back into the highschool classroom.

A New Course

The idea is to create a new course with the working title *Geometric Reasoning* after which preservice teachers have acquired – besides geometric knowledge – mathematical language skills, and a network of connections between highscool and university mathematics. This network together with a growing desire for justification should have them well prepared to start their Let-ε>o-Analysis and their Let-k-be-a-field-Linear Algebra courses together with mathematics majors.

Following the outlined goals, the contents of the course should start with specific high-school problems and develop them into formal and abstract proof based mathematics, adopting the strategy to preserve a continuity in the mathematical biography (Beutelspacher, A., Danckwerts R., Nickel G., Spies S., & Wickel G, 2011). At the same time, the course should intentionally leave open ends, pointing out where additional mathematics is needed to tackle the problem. Geometry seems particularly well suited for this kind of two-sided docking. For an instance, start with ruler and compass constructions, leaving an open end to (Linear) Algebra at the impossibility to construct the regular 7-gon. Limit processes in volume computations provide a link to Analysis.

It is imperative that the course starts to develop the students' ability to generalize, and to work with abstract concepts such as the axioms of incidence geometry. The students need to learn the language of university mathematics, the oral and written communication of simple mathematical arguments and proofs – all at a manageable level. Geometry has a lot of examples to challenge misconceptions and to refine intuition. It should not be hard to substantiate the relevance of this Geometry for high-school mathematics such as Schwarz & Herrmann (2015) did for Linear Algebra.

The New Curriculum

Semester	Major	Minor
1	Geometric Reasoning (10 CP)	Geometric Reasoning (10 CP)
2	Linear Algebra 1 (10 CP)	Linear Algebra 1 (10 CP)
3	Linear Algebra 2 (10 CP) Mathematics & Computers (5 CP)	Analysis 1 (10 CP)
4	Analysis 1 (10 CP) Stochastics (10 CP)	Stochastics (10 CP)
5	Analysis 2 (10 CP) Algebra & Number Theory (10 CP)	Algebra & Number Theory (10 CP)
6	Seminar (5 CP) Bachelor's Thesis (10 CP)	Seminar (5 CP) Mathematics & Computers (5 CP)

Resources

Part of the appeal of the present concept to the practitioner is that it might actually materialize. It is minimally invasive in that it does not stir up the entire mathematics major curriculum. Reuniting mathematics majors and preservice teachers in Analysis and Linear Algebra frees capacities for separate Algebra and/or Stochastics classes.

In order to create a sustainable model, it is planned to set up an instructor wiki which in a first step should contain a script for the lectures, a reference list and a growing collection of exercises.

Conclusion & Outlook

This is an implementation project, not an actual research project. (Yet?) For one, I hope for input from researchers. But, in the best of all scenarios, the project will be evaluated/accompanied with feedback going both ways. A scientific discipline should not lose out of sight the transfer of its results into practice. Unfortunately, we do not have the time and resources to spend a decade on a design based approach as reported by Larsen, Johnson & Bartlo (2013) for instructional innovations teaching group theory. But certain aspects of their work, suitably scaled, should be applicable here.

The measures proposed here address mainly the first discontinuity in the biography of preservice teachers. At the department, we are also discussing a capstone course involving both active high school teachers and finishing teacher students, developing the approach by Winsløw & Grønbæk (2014). Here, the implementation challenges will be even more complex as it involves yet another administration.

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Why linear algebra is difficult for many beginners

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Linear algebra is a fundamental mathematical discipline, which provides a basic language with a wide range of application. It was developed from different phenomenological roots (algebra, geometry) towards an abstract axiomatic theory. This is the main reason why linear algebra is difficult for beginners, at least for those who have problems to understand mathematics mainly by structure sense.

Phenomenological roots

The historical development of linear algebra was initiated by two different constraints, namely the problem of solving linear equations and the desire for numerical description of geometrical objects. The latter was due to a vision of Leibniz to make the analysis of location and movement in geometry accessible to manipulation by algebraic formulas.

The resulting reflections developed towards the theory of vector spaces and linear mappings as a common superstructure. "The concept of vector space, so elementary in terms of structure, encapsulates, in a very elaborated product, the result of a long and complex process of generalization and unification." (Dorier 2000, p. 59) Therefore linear algebra is a highly demanding domain of mathematics from the cognitive point of view.

Levels of abstraction

The concept of vector space is the result of an expanded "praxeological progression" (Winsløw 2014) with successive levels of abstraction. The concept of a vector itself is a common abstraction of already abstract objects like the sequence of coefficients of a linear equation or geometrical translation (Dorier & Sierpinska 2001), whereas „most concepts in school mathematics can be traced back to an origin in material physical activities of some sort or another (such as counting, measuring, drawing, constructing)." (Dörfler 2003, S. 154). To grasp these elaborated concepts requires sophisticated mental activities such as "encapsulation" (Dubinsky & Harel 1992), "objectivation" (Radford 2010) or "reification" (Sfard 1995, 2000). This also applies to advanced concepts like factor space and dual space.

The axiomatic approach

The development of mathematics towards a theory of abstract mental concepts has an exterior counterpart in the current tools of representation. The professional mathematical language is a highly developed artefact, usually related to the semantic of sets, which often means that other semantic implications or phenomenological roots are hidden and have to be revealed by the learners. So beginners in a course of linear algebra often have no orientation to what needs the theory responds, whereas the lecturers act within a long-range-perspective, which is not shared by the learners. In addition the strong logical hierarchy of

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the axiomatic setting requires definitions and proofs, which do not seem necessary to an original point of view. Why for example is the following claim worth of special consideration: In a vector space over a field K with unit element 1_K the equation $1_K x = x$ is valid for each vector x.

Symbol sense and mental flexibility

Further the mathematical language is highly conventionalised and it causes a high density of information, which requires symbol sense and a specific reading ability. In an expression like $M_{n,m}(K)$ each sign together with its position in the sequence of signs mediates a specific information which has to be decoded. Definitions are often formulated in a manner that fits operational needs but conceals the original meaning (for example the usual definition of linear independence).

In addition the ability of flexible interpretation of the assigned objects within the theory is needed. A matrix can be considered as a system of scalars, a system of row respective column vectors or as a vector itself and it depends on the context which interpretation is appropriate. This ability is also needed for applications of the theory, for example when functions have to be considered as elements of a vector space.

The theory on the whole requires flexible changes between different points of view, especially between calculation and structural considerations.

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Online tests for evaluating learning success

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In a pilot project online tests with "unusual" multiple-choice questions covering some of the topics of "Higher Mathematics A for Electrical Engineers" have been developed and used in a test run in winter semester 2015/16. This paper gives the mathematics education background and discusses the different types of multiple-choice questions developed in the project and their educational scope. Problems with low participation rates in the test run as well as directions for future research are also discussed.

Introduction and background

In a working group concerned with mathematics education for engineers the author proposed a project where multiple-choice questions testing understanding rather than schematic application of mathematical tools should be developed for the different course topics of "Higher Mathematics A for Electrical Engineers" ("Höhere Mathematik A für Elektrotechniker") at Paderborn University. The ensuing discussion led to the idea of having online tests that can be taken by the students at any time within a certain time frame after the topic has been completed in the course. Two colleagues from engineering teaching the introductory engineering courses in the first semester and one colleague from mathematics who had just taught the introductory mathematics course for the mechanical engineering students expressed interest in joining the project with their own courses. Subsequently, it was agreed that the author would make a test run in a pilot project for some selected topics from "Higher Mathematics A for Electrical Engineers" in winter semester 2015/16.

By now 95 questions have been developed and implemented with Moodle for the test run which is open to electrical engineering students as well as mechanical engineering students und physicists, in order to get a larger cohort of students in the test run.

Multiple-choice questions have been used for a long time and are one of the most popular assessment tools in education (Hassmén & Hunt, 1994, p. 149), since they are easy to implement and mark (and can even be marked electronically without any involvement of the teacher). Wood (1977) gives a comprehensive overview of multiple-choice testing, which, of course, does not yet reflect any of the possibilities offered by computer-aided assessment. Criticism has been voiced that multiple-choice questions are not suitable for testing a deeper understanding, since they offer a set of predetermined answers and usually do not ask the learner to do her/his own problem solving without any a priori information about the possible answers. See e.g. Hoffmann (1978) for a rather passionate criticism of multiple-choice testing. Various authors have given guidelines for designing multiple-choice questions (see (Hansen & Dexter, 1997) and the references therein), and it has been investigated whether additional self-assessment (see e.g. (Hassmén & Hunt, 1994)) or/and partial scoring (see e.g. (Kansup & Hakstian, 1975)) can improve the reliability of multiple-choice testing. The effect of students' guessing and how this can be remedied has also been considered (see e.g. (Wood, 1976)). The given references on multiple-choice testing are far from complete and serve only as exemplary references to set the general context for this project.

In the context of testing or assessment, particularly with large numbers of students, there has been an ongoing discussion (since at least the 1970s) on whether assessment with multiple-choice items can test the same assessments components as constructed-response items. See e.g. the publications by Kamps & van Lint (1975), Traub & Fisher (1977), Katz, Friedman, Bennett & Berger (1996) and Huntley, Engelbrecht & Harding (2009b). Rodriguez (2003) analyses 67 empirical studies concerned with the trait or construct equivalence of multiple-choice items and constructed-response items. He draws the conclusion that "Construct equivalence, in part, appears to be a function of the item design method or the item writer's intent." (Rodriguez, 2003, p. 163). Katz, Friedman, Bennett & Berger (1996) give interesting insights into high school students' approaches to multiple-choice as well as constructed-response questions, and Kamps & van Lint (1975) and Huntley, Engelbrecht & Harding (2009b) provide evidence that certain assessment components in basic mathematics courses at university level can very well be tested with multiple-choice questions. Again the references given are far from complete but serve as exemplary references for this context.

In the project described above we are interested in an "unusual" type of multiple-choice questions. The questions are "unusual" in that they explicitly test understanding by asking the student to pick the one right (or one false) statement among five given statements concerned with a certain course topic. Finding the one correct (or one false) statement will, in most cases, require that the student does indeed solve several smaller problems. The nature of these questions is somewhat similar to the more sophisticated biology questions reported by Tamir (1993, see e.g. Exhibit 4), and there are both positive and negative questions/items (Tamir, 1993) in the online tests. Kamps & van Lint (1975, in particular items H and J of their multiple-choice test) and Huntley, Engelbrecht & Harding (2009b) give some examples of mathematics assessment questions that go in a similar direction. Variants of such multiple choice questions appear to have been used in written examinations at some German universities, and Schmidt, Macht, & Hess (2005) provide a collection of mathematical multiple-choice questions with detailed answers, predominantly taken from mathematics examinations for economics students. The questions published by Schmidt, Macht & Hess (2005) are of similar types to those developed by the author, but the average level of the questions is lower and they do not cover all the types of multiple-choice questions developed in the pilot project. In particular, the multiple-choice questions concerned with abstract statements (see below) are missing in Schmidt, Macht & Hess (2005). Some of the different types of the multiple-choice questions developed by the author will be discussed below and will be illustrated by four sample questions. These multiple-choice questions must also be seen as mathematical exercises. For a recent survey and background reference on the use, design and effects of exercises refer to Leuders (2015).

The types of questions in the online tests

In the pilot project 95 multiple-choice questions have been developed covering the first five topics of "Higher Mathematics A for Electrical Engineers": 1. Sets and functions, 2. Vector calculus, 3. Linear systems of equations, 4. Further foundations, 5. Sequences of real numbers. Each multiple-choice question offers five statements and asks the student to select the one correct (positive questions) or one false (negative question) statement. The student must choose exactly one statement, and due to the nature of mathematics (and careful wording) there will be exactly one correct or one false statement, respectively, among the five given statements. So the ambiguity of what is the “best” answer (for positive questions) that has been criticized with respect to multiple-choice (see e.g. (Hoffmann, 1978)) should not occur in this context.

Besides positive and negative questions there are two basic types of questions: those that deal with examples and those that deal with abstract statements about the course content. Within each of these types the five given statements may either cover all exactly one topic (e.g. convergence of sequences of real numbers) or they may spread wider within one larger topic of the course (e.g. statements about rather elementary properties of vectors). In either case, the answer will not be obvious, and even a good student will usually have to sit down and work out (in her/his head or with pen and paper) which one of the given five statements is the correct one or the false one. Thus the multiple-choice questions require problem solving on the student’s part, and some of the ones dealing with abstract statements about the course content will even require that students draw their own conclusions and make up their own examples.

Before we come to the sample questions, some words should be said about when the online tests shall be taken, what feedback the students will receive and what the intended learning outcomes are. Separate online tests will be taken for each course topic once the topic has been completely covered including the tutorial classes for this course topic. So all students should have done the exercises from the corresponding exercise sheets and should be familiar with the topic and with standard exercises for this topic. Since the assessment regulations do not allow us to make participation in the online tests compulsory, these can only be advertised and recommended as an additional offer to get feedback on one’s own performance with respect to the most recent topic of the course. The students can do the online tests from any computer at any time (while the test is still open) and they can use any resources they like for help. The aim of the online tests is to provide the students with feedback on their learning and to help them with their learning. It is expected that good students will enjoy the different format and will be challenged to view the course material from yet a different angle, while week students will receive a timely feedback if they need to invest more time in studying and doing the exercise sheets. It is especially hoped that week and struggling students will thereby realize the need to study more (and often differently with better learning strategies) early on in the course, in contrast to having already been completely lost when this realization only comes at the final examination. After completing each multiple-choice question the students will see a feedback text that addresses the relevant points for solving this question. Not all steps will be given but the relevant points, so that students who did not know how to handle the question can now work out the details with this help. At the end of the multiple-choice test the students will get an overall score and some general feedback on their performance.

Four samples of multiple-choice questions are shown below, three without the feedback text, and one with feedback text presented as screen shots from the online implementation in Moodle. It should perhaps be mentioned here, that the course "Higher Mathematics A for Electrical Engineers" is taught in German; so the samples of multiple-choice questions below as well as the feedback text have been translated by the author for the express purpose of this paper.

Sample 1: Which of the following statements about the complex numbers $z = 2 + 4j$ and $w = 2 - 2j$ is false? All but one of the statements are correct.

(a) $z + w = 4 + 2j$

(b) $\frac{z}{w} = -\frac{1}{2} + \frac{3}{2}j$

(c) $|z| = 2\sqrt{5}$ and $|w| = 2\sqrt{2}$

(d) $\bar{z} = 2 - 4j$ and $\bar{w} = 2 + 2j$

(e) $z \cdot w = 4 - 8j$

This question covers a variety of different operations for complex numbers for an example and is a negative question (i.e. the one false statement, here (e), must be identified). In order to solve the problem, the student must work her/his way through the statements, computing each given object until she/he finds the false statement. It will be explained to the students at the beginning of the online test that it is recommended that they look at all statements (even if they believe that they have already identified the correct/false one) for the learning benefit. This is a rather easy question that the majority of the students will probably answer correctly. However, it seems advisable to include some rather easy questions at the beginning of and throughout each online test for motivational purposes.

Sample 2: Which among the subsequent sequences of real numbers is convergent? All but one of these sequences of real numbers are divergent.

(a) $\left(1 - \frac{1}{n^2} + \sin\left(\frac{1}{n}\right)\right)_{n\geq 1}$

(b) $\left(2^n - n^2\right)_{n\geq 0}$

(c) $\left((-1)^n\right)_{n\geq 0}$

(d) $\left(e^{n^2}\right)_{n\geq 0}$

(e) $\left(\ln(n)\right)_{n\geq 1}$

This question may be seen as a positive question, since the property convergence is only satisfied for one of the examples. Here the student needs to inspect the different examples to find the convergent sequence. If a student knows that the sequence in (c) is divergent and that unbounded sequences cannot be convergent, then the sequence in (a) is easily identified as the convergent one with limit 1.

Sample 3: Which of the following statements is false? All but one statement are correct.

(a) The sum of two divergent sequences is also a divergent sequence.

(b) The limit of a convergent sequence is uniquely determined.

(c) The product of a bounded sequence and a sequence converging to zero is a convergent sequence.

(d) If a sequence is monotone and bounded, then it is convergent.

(e) Every convergent sequence is bounded.

This question deals with abstract statements from the topic convergence of sequences of real numbers and is one of the most challenging questions. This question tests in parts familiarity with standard results from the topic sequences (statements (b), (d) and (e)), whereas (c) will have been stated differently in the course, noting that the product of a bounded sequence and a sequence converging to zero also converges to zero. The false statement (a) however will seem plausible to most students, and only a suitable counterexample will reveal why this is not correct.

Sample 4: The fourth sample is shown as a screenshot from Moodle, where both the question and the feedback text are shown. In the screenshot below an incorrect selection has been made (indicated by the red cross, whereas a correct selection would be indicated by a green tic-mark). After making this selection and submitting, the correctness of the selection is indicated as shown and the explanation in the second screenshot below is always displayed (regardless of whether a correct or an incorrect selection was made).

Question 1

Incorrect

Mark -0.25 out of 1.00

Which of the following statements about the function

$$f:\mathbb{R}\to\mathbb{R},\qquad f(x)=\begin{cases}\dfrac{17}{x+1}, & x\neq -1,\\ 0, & x=-1,\end{cases}$$

is **correct**? Only one answer ist correct.

Select one:

- **a. The function is mathematically not correctly defined.**
- **b. The function is surjective but not bijective.**
- **c. The function is bijective.**
- **d. The function is strictly monotonic.** ✗
- **e. The function is injective but not surjective.**

The selection is incorrect.

Explanation: The function is mathematically correctly defined; therefore the statement "The function is mathematically not correctly defined." is wrong.

The correct statement is "The function is bijective.". Indeed for $x>-1$ the values of the function cover all positive real numbers, $f(-1)=0$, and for $x<-1$ the values of the function cover all negative real numbers. Hence, the image is $\mathbb{R}$ and f is surjective. From

$$\frac{17}{x+1}=y \iff \frac{17}{y}=x+1 \iff \frac{17}{y}-1=x$$

we see that the equation $f(x)=y$ has exactly one solution in $\mathbb{R}$. Hence f is also injective. Thus f is bijective.

Why are the other statements wrong?

- $f(-18)=-1>f(-9{,}5)=-2<f(-1)=0$ but $-18<-9{,}5<0$ implies that f cannot be strictly monotonic. Thus the statement "The function is strictly monotonic." is wrong.
- The statements "The function is injective but not surjective." and "The function is surjective but not bijective." cannot be correct, since f is bijective.

This is a positive question of the type where several of the new mathematical concepts are considered for one particular example. Piecewise defined functions appear to be unknown to most students from their school mathematics, and the new notions of injective, surjective and bijective are always challenging. So, although this question deals with a very concrete example, it is expected that most students will find this question far from easy.

Low participation rates in the test run

After implementing the 95 questions in Moodle, the author discussed the project with the colleagues who teach the first year mathematics courses for the electrical engineers, mechanical engineers and physicists in winter semester 2015/16, and all three colleagues agreed to support the project by recommending it to their students. They also invited the author into one of their lectures to promote the project in person by explaining what the project is about and showing how to get a Moodle account as well as giving a demonstration with some questions from the first online test. Despite this, the participation rate was quite low. In fact, so few students have actually completed the tests, that analyzing the statistics for each test will not give any useful insights. In the test run, the online tests have not been closed after a certain time period, and are still open during the academic year 2015/16. While it is possible that some more students may attempt the online tests during their exam preparation, it is clear that something needs to be done to improve the participation rates in the online tests.

Directions for future development and research

On the development side, it is clear that the existing 95 questions have to be reviewed and improved. Furthermore, questions for the remaining five course topics need to be developed: 6. Continuity of real-valued functions, 7. Differentiability, 8. Integration, 9. Ordinary differential equations, 10. Infinite series. The author expects to develop another 150 questions for these topics. There would also be the possibility to add a second or even third set of some standard questions to the existing tests (and also to the ones to be still developed) such that students can have another try or two at these types of questions if they answer these types of questions incorrectly the first time. Likewise it would be possible to create some sets of specific types of standard questions so that each student gets a random sample from these types of questions.

Should the development and implementation of these questions be ready in time for a first full run in winter semester 2016/17, serious thought needs to be given how to get a better participation rate in the tests, so that the students can actually benefit from this additional offer to get feedback via informal formative assessment through the online tests. As mentioned earlier, the current examination regulations make it impossible to make participation in the online tests compulsory, which makes it difficult to motivate students to take these tests. However, in winter semester 2016/17 the author will teach "Higher Mathematics A for Electrical Engineers" herself, and will thus have a much better opportunity to advertise and promote the online tests than this was possible in the test run. It may also be possible to give a bonus for participation in the online tests, which will allow students to improve their overall mark in the course if they have passed the final examination. We know from past experience that a bonus will greatly increase participation rates. However, since we cannot prevent students from getting help when they do the online tests, such a bonus must, on the one hand, be small enough but, on the other hand, be still attractive enough to encourage participation.

What are the directions for future research beyond the development and improvement of the questions for all ten topics of "Higher Mathematics A for Electrical Engineers"? There are two main avenues of research that the author is interested in pursuing:

1) A detailed classification of the questions into different types

The classification of the questions has already been briefly discussed in the second section of this paper. It is clear that a proper classification needs to be much more detailed and comprehensive and will very likely end up with more than the basic types of questions that have been identified so far. Furthermore it may also be interesting to consider certain types of questions in the context of individual topics. Generally the questions from the online tests should also be classified with respect to a suitable taxonomy for mathematics, e.g. the assessment component taxonomy proposed by Huntley, Engelbrecht and Harding (2009a, 2009b). For more literature on assessment models and taxonomies in mathematics see also the works by Bloom (1956), Niss (1993) and Smith, Wood, Crawford, Coupland, Ball & Stephenson (1996).

2) An investigation of the misconceptions covered by the different questions

The author has over ten years of teaching experience and, from her professional practice, she is therefore quite familiar with many misconceptions that students may develop when they study higher mathematics at university. When the author taught "Higher Mathematics A for Electrical Engineers" in winter semester 2014/15, she personally revised all existing exercise sheets and solutions and added several questions and solutions to the exercise sheets from the previous run, led the large problem class in the lecture hall and also gave one of the small group tutorial classes. This experience provided a wealth of insight into the students' problems with the course material and the misconceptions about each topic. It seems interesting to analyze a representative set of questions from each course topic to underpin their design with the currently available literature on student's misconceptions about the respective topic. For more advanced topics or concepts the author expects that the literature on misconceptions within these topics or concepts may not be very extensive. It might also be interesting to contribute to the literature on students' misconceptions should there be any topics where so far little or nothing has been published about misconceptions within these topics.

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Mathematics support for non-maths majors: A senior management perspective

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For several years it has been well-known that many students taking non-maths majors are challenged by the quantitative elements of their courses. Mathematics Support was developed in the 1990s as a means of assisting such students. This paper presents an overview of the key findings from a qualitative research study involving senior managers in a wide range of higher education institutions in England in relation to the needs of their students in mathematics and statistics. These findings indicate that all universities have a significant number of students who need additional support in mathematics and/or statistics if they are to be successful in their studies and beyond.

What is Mathematics Support and why is it needed?

Many disciplines are becoming increasingly quantitative – not only disciplines such as engineering, the physical sciences and economics, which have long relied on mathematical models, but also, more recently, subjects such as the biosciences, psychology and many social sciences. In England, the vast majority of students study no mathematics after the age of 16. For many of these students who select non-mathematics majors in higher education, the need to re-engage with mathematical or quantitative methods can be challenging and they are often ill-prepared for this challenge. A report by the Advisory Committee on Mathematics Education estimated that, each year, around 330,000 students enter courses in higher education where it would be beneficial for them to have studied mathematics beyond GCSE (the national qualification taken at age 16)[i] whereas only 125,000 have actually done so (ACME, 2011).

In response to the needs of their students, many universities have introduced some form of Mathematics Support (MS). The most common form of MS is the 'drop-in' centre – a staffed location where students attend (if they wish) at a time of their own choosing and receive one-to-one support in relation to those areas of mathematics that are causing them the most difficulty. MS is typically provided in addition to the regular lectures, seminars, tutorials, etc. that make up the 'normal' teaching on a course and is accessed voluntarily by those who perceive they will benefit from it.

The development of Mathematics Support in the UK

Large-scale MS in higher education in the UK began in the early 1990s – the BP Mathematics Centre at Coventry Polytechnic established in 1991 was one of the first permanent drop-in centres. During the 1990s a number of other centres were opened. However, Kyle (2010, p.103) described the early days of MS as "a form of cottage industry practised by a few well

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meaning, possibly eccentric individuals". In these early days, MS was usually a grass roots initiative introduced by mathematics lecturers seeking to remedy the under-preparedness of many of their students.

In 2005, **sigma** a collaboration between the MS provision at Loughborough and Coventry Universities was designated by the Higher Education Funding Council for England (HEFCE) as a Centre for Excellence in Teaching and Learning (CETL). This not only bestowed status on MS but also a substantial amount of funding and during the 5-year funding period, **sigma** was successful in establishing MS as a mainstream activity in many universities – to the extent that Kyle's article (*ibid*, p.104) concludes that "Mathematics support came of age in the first decade of the 21st century". Following this, **sigma** was commission by the National HE STEM Programme[1] to undertake further work promoting MS in the HE sector. In **sigma**'s final report for this Programme, David Youdan, Executive Director of the Institute of Mathematics and Its Applications, is quoted as follows:

> *"It is hard to overstate the importance of the expansion of the* ***sigma*** *network ... The accepted position is that it is now a student's right to receive support with the mathematical content of their degree." (Fletcher, 2013, p.49)*

Qualitative Study: Senior Management Perspectives

As previously stated, MS began as a grass roots initiative, but as it demonstrated its value to universities it became more of a main-stream activity. In 2013, **sigma** received further funding from HEFCE to establish a sustainable community of MS practitioners. As part of this work, **sigma** engaged with senior managers in universities to seek to gain an understanding from their perspective of the needs of their students in terms of MS. To our knowledge, this is the first piece of work which has sought to understand the senior management perspective on the provision of MS. In order to gain an in-depth understanding, a qualitative approach was used to gather the views of senior managers.

Semi-structured interviews were carried out with senior managers, typically at the level of Pro-Vice-Chancellor (PVC) for Learning and Teaching in 23 universities from across the spectrum of higher education (including large research intensive, smaller research focused and newer more teaching focused institutions). The interviews explored issues such as the challenges faced by students in relation to mathematics and statistics, how those challenges are being addressed, the degree to which MS is embedded in the institution, any plans to further develop MS and external support that universities felt might be helpful in terms of supporting their students mathematical and statistical needs.

Full details of the qualitative study and an extensive analysis of its outcomes can be found in Tolley and Mackenzie (2015). The following section gives a brief precis of some of the key findings.

[1] STEM stands for Science, Technology, Engineering and Mathematics; the National HE STEM Programme (http://www.hestem.ac.uk/) was a major HEFCE funded programme to promote entry to and success in STEM disciplines in Higher Education.

Key findings and future developments

All 23 interviewees reported that there are students at their institution who are challenged by mathematics and/or statistics. This was attributed to a number of factors including the small proportion of students who study mathematics post-16, the increase in inhomogeneity of undergraduate cohorts that has taken place as the sector has grown and the negative attitudes towards mathematics and statistics that many students have developed during their pre-university education. In addition, many interviewees reported that, even where students had achieved a good grade in A Level Mathematics (the final pre-university qualifications taken at age 18), many had difficulty in applying the knowledge they had gained to solve problems in unfamiliar settings (typically in the context of their main discipline of study).

The comments above did not only relate to students studying STEM disciplines, but rather applied to a rapidly expanding range of subjects that make increasing use of quantitative methods and mathematical models. The comments also did not only apply to undergraduates, but in many cases were more focused on postgraduates. In some disciplines, postgraduate study is considerably more quantitative than undergraduate study and bachelors' degrees do not adequately prepare students for the more quantitative approach to the discipline at postgraduate level.

All of the interviewees recognised that unless appropriate forms of learning support in mathematics and statistics are provided then it is inevitable that there will be an adverse impact on their students' satisfaction, retention, achievement and employability. As a consequence, in many institutions decisions about the provision of MS are increasingly not being left to individuals at the grass roots level to take in isolation, but rather are becoming part of wider strategic considerations with MS being seen as part of a range of institutional support provision.

Some institutions reported that they were considering requiring students to have achieved an A level in mathematics in order to gain entry to courses which previously had not had a specific mathematics requirement beyond the general matriculation requirement (applying to all courses in the institution) of GCSE Grade C. Whilst such an approach should mean the students that are enrolled are better prepared, the danger of such an approach is that the pool of students who have taken A level mathematics is relatively small and such a decision is likely to have an adverse an impact on institutional recruitment.

Many interviewees identified professional development for their staff as a key need. Two distinct groups of staff with different needs were identified. Firstly, the needs of specialist staff working in mathematics and statistics support were recognised. In many institutions there may be only one such specialist and so the importance of networks for such staff to share good practice were highlighted. This is one of the things that **sigma** is seeking to provide as it develops a sustainable community of MS practitioners. Senior managers suggested that in addition to appropriate training, some kind of recognised or professionally accredited status would be beneficial.

The second professional development focus that was identified related to the embedding of mathematics and statistics support in modules within the non-maths disciplines. Many senior managers acknowledged that many of their staff in these disciplines themselves had issues of confidence and competence in relation to mathematics and statistics. This confirms the findings of the RSA report *Solving the maths problem* (Norris, 2012) which asserted that "English universities are side-lining quantitative and mathematical content because students and **staff** lack the requisite confidence and ability" (p.11, my emphasis). Senior managers believed that professional development was needed for colleagues in these disciplines in terms of their own quantitative and mathematical skills and also in terms of designing appropriate curricula and teaching materials.

The participants in the study also identified significant benefits from institutional networking in relation to MS. It was recognised that many universities faced the same issues and so there were major efficiencies to be gained through the sharing of resources, particularly those focused at the stage of transition from school/college into university. Many of the senior managers were aware of the work of **sigma** and the resources that it provides through the **math**centre and **stats**tutor websites (www.mathcentre.ac.uk and www.statstutor.ac.uk). A number of PVCs referred to **sigma** as the 'go to' organisation for information about MS.

Overall the findings from this study show that senior managers are well aware of the difficulties of their students in relation to mathematics and statistics. They further understand that this situation is unlikely to change significantly in the near future and that, consequently, the need for MS provision will remain. In addition, they recognise the benefits of belonging to a specialist network that can provide professional development opportunities for staff at a local level.

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[i] In England, all school students are required to study for GCSE mathematics – a qualification that is typically taken at age 16. Further study of mathematics after this is optional. Those who intend to progress to university to study a mathematics-rich subject will study A level mathematics – this is a two year course with the final examination usually taken at age 18. Although mathematics is currently the most popular A level subject, the vast majority of the cohort do not study A level mathematics, or any other mathematics, post-16.

Proof-oriented tutoring: A small group culture utilising research strategies of mathematicians

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Our observational study of 26 lecturers' mathematics teaching in a small group tutorial context drew on a grounded analytical approach to analyse the different practices lecturers employ to teach mathematics. An in-depth study of a tutoring culture, formed by one of these lecturers, revealed that the majority of lecturer and students' time was used for proof. Observation and interview data provided evidence for a number of the lecturer's strategies for teaching proof. Findings indicated that the lecturer's strategies from her research, such as a variety of heuristics for thought processes, informed her strategies for teaching proof. In this paper, we exemplify one teaching strategy.

Introduction

A review of literature with regard to proof and proving reveals a research focus on the roles of proof in mathematics scholarship and possible implications for teaching. Central roles of proof within mathematics are reported as: *verification* (to establish truth through proof), *explanation* (to gain insight into why it is true), *systematisation* (to organise results into a deductive system), *discovery* (to discover new results), *communication* (to convey mathematical knowledge), and *incorporation* of well-known results into new frameworks (Bell, 1976; de Villiers, 1990; Hanna and Jahnke, 1996).

Reflecting on roles of proof, Bell (1976), de Villiers (1990) and Schoenfeld (1994) asserted that students should experience all roles of mathematical proof and proving, as mathematicians do. Based on this assertion, Schoenfeld (1994) shared his experiences for teaching proof to undergraduate students, and through that, his teaching of what it is to do mathematics and think. Problem solving heuristics illustrated in proofs were highlighted. Hersh (1993) and Hanna and Jahnke (1996) offered another perspective according to which differences between proof for research and proof for teaching occur. For instance, Hersh (1993, p. 396-397) stressed that "[s]tudents are all too easily convinced", so the role of proof in teaching is not *verification*; it is instead "admission into the catalog of primarily absolute truths" or *explanation*. However, not all of the aforementioned scholars based their exposition on empirical studies of teaching in lecturing or alternative contexts.

Through interviews with lecturers, Weber (2010) and Yopp (2011) investigated the roles of proof in university teaching. Yopp's (2011) results indicated that in proof for teaching, *discovery* and *communication* were absent from 14 research mathematicians and statisticians' sayings. Also, Weber (2010) stressed the lecturers' lack of strategies to achieve their goals in teaching proof. However, in neither study was the lecturers' actual teaching observed. Our observational study investigates the practices lecturers employ to teach mathematics in

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a tutorial context. In this paper, we draw on our study to present a lecturer's actual teaching of proof, and shed light on the roles of proof in her teaching.

Methodology

At an English university, students were expected to attend lectures (100+ students) and tutorials (2-8 students) for the first year of their mathematics degree. Tutorials were 50 minute weekly sessions provided for work on lecture materials, usually in analysis and linear algebra. Zenobia was the tutor for a small group of five students. She was an experienced lecturer in modules offered by the mathematics department and a researcher in analysis. The tutorial group included four high achieving students and an average student.

In our study, observational and interview data of 26 lecturers' teaching in tutorials was audio-recorded and transcribed. Subsequently, data was gathered systematically from tutorials of Zenobia and two other of the 26 lecturers for more than one semester. Through a grounded approach to the data, an analytical framework *TKiP* (*Teaching Knowledge-in-Practice*) emerged. In this paper, we use only the category *teaching practice* of *TKiP*.

In *TKiP*, the unit of analysis for practice is the *tool*. We drew on Vygotsky's *tools* (1978) and saw a *tool* as carrying its material and psychological nature inherently (Cole, 1994). For instance, a graph of an injective function is a *tool* for teaching injectivity since it might suggest meaning for injectivity (psychological nature). A graph, however, is a curve drawn in a material whiteboard or page (material nature). In the context of *tools*, lecturers' actions in teaching were seen to be *modes of action with tools* (given the name *strategies*). For instance, a lecturer's use of graphs can be seen as a *strategy*. In case a *strategy* is identified repeatedly in the data from a lecturer, this *strategy* forms a *characteristic* of this lecturer's teaching. So, teaching practice in *TKiP* is analysed into *tools*, *strategies* and *characteristics* of teaching. An in-depth study of Zenobia's teaching revealed that the majority of Zenobia and students' time was used for proof. We took a grounded analytical approach to analyse Zenobia's practice of teaching proof in her tutorials. In order to present Zenobia's teaching of proof, we identified her teaching *tools* and *characteristics*. *Characteristics* of teaching, as repeated *strategies*, enabled us to gain insight into the culture formed in Zenobia's tutorials.

Zenobia's teaching of proof

Zenobia often used the whole tutorial time for one proof. In observations, she informed students that: "Most of proofs you will see this year is that something satisfies a certain definition." The definitions for the tutorial proofs were in linear algebra (subspace, bijective linear transformation) and analysis (unbounded sequence, ε-δ definition of limit, convergence of sequence, subsequence, bijective function). A teaching goal she declared in interviews was that: students should pass the exams. Zenobia also reflected on a second goal in interviews: students should work on fundamental topics (for instance, in analysis) and experience 'what it feels to understand mathematics'. We looked at Zenobia's actual teaching practice to make sense of 'what it feels to understand mathematics' for her. A list of Zenobia's *characteristics* related to teaching proof and mathematical thinking is: *selection of tasks*, *dissecting definitions*, *creation of students' positive feelings and participation*, *public presentation of results*, *getting intuition by visual reference/reference to examples* and *evaluation of the* *students' sense making of mathematics*. In this paper, we exemplify one *characteristic*: *getting intuition by visual reference/reference to examples.*

In interviews, Zenobia explained to us three steps she implements for conducting her mathematical research, and which, she declared, she also uses in her teaching.

> *The first step [in doing research] is the decoding where you are given a problem and you have to understand what the problem is, what everything mean [e.g. by experimenting with images against definition], why it is a problem; the second step is with this picture that you have got from the decoding process, you get some intuition, you play around with things in your head a little bit and then you get this sort of 'aha I figured it out, I have got this idea now of why that works' and then you have got the encoding process [i.e. the third step] where you write it down [formally].*

Our observations provided us with evidence for Zenobia's declared connection between her teaching practice and her own mathematical research practice. In addition, we asked Zenobia to reflect on evidence for her declaration in tutorials we observed. We made sense of these connections through our analysis of *tools* and *strategies* for the three steps in her teaching and research. In the above quotation, we coded the extracts "understand what the problem is", "what everything mean [e.g. by experimenting with images against definition]", "why it is a problem" and "get some intuition" as declared research *strategies*. In brief, these research *strategies* were heuristics for thought processes. Taking a grounded approach, we coded extracts of transcripts of her teaching as teaching *strategies*. *Getting intuition by visual reference/reference to examples* is one of Zenobia's teaching *strategies* which were identified repeatedly in the data. This *strategy* was considered to form a *characteristic* of Zenobia's teaching, which we now exemplify. For instance, the proof task in tutorial number 8 was:

If s_n converges to l, then every subsequence of s_n also converges to l.

A student was in charge of writing on the board. Students contributed for the writing up of the definition of convergence on the board:

A sequence s_n converges to l if $\forall\ \varepsilon > 0\ \exists\ k_0 \in R$ s.t. $|s_k - l| < \varepsilon\ \forall\ k > k_0$.

Zenobia asked the student on the board to sketch a convergent sequence. The student sketched a graph of a sequence which converges to 0. Zenobia then asked all students to put values of s_n, ε, and k_0 on the graph. Another student was invited to the board to select a subsequence on the graph and define it. The students contributed so that the student on the board wrote:

A subsequence of a sequence s_n is a new sequence $r_m = s_{f(m)}$ where f is an increasing function $f: N \to N$.

Zenobia then asked all students to put values of r_m on the graph. She wrote on the board that they know the two aforementioned definitions and they need to prove:

$$\forall\ \varepsilon > 0\ \exists\ \hat{k}_0 > 0 \text{ s.t. } m > \hat{k}_0 \Rightarrow |r_m - l| < \varepsilon\,.$$

A students' key result was to get $\hat{k}_0$ from k_0. Students were experimenting with special cases for $\hat{k}_0$ and k_0 on the graph. From special cases, they generalised that: Since f is strictly increasing, $\exists\ \hat{k}_0\ \ f(\hat{k}_0) > k_0$. For $\mathrm{m} > \hat{k}_0$, $|\, r_m$- $l| = |\, s_{f(m)}$- $l| < \varepsilon$ because

$$f(m) > f(\hat{k}_0) > k_0.$$

In interview after the tutorial, Zenobia reflected on the use of the graph. She said:

> *I think what I am trying to do here is to get them to connect because they have an intuition. This intuitively isn't obvious for everyone, so the point is to understand how your intuition about convergence connects to the rigorous definition of convergence.*

We coded the transcript extracts regarding work on the graph as Zenobia's *strategy*: *getting intuition by visual reference/reference to examples*. According to the above quotation, the students would get *intuition* for the definition of convergence. Considering that notation of the definition was negotiated on the board, intuition about convergence was potentially a shared intuition among students. The *reference* was *visual* since it was a *graphical representation*, and in this instance, this representation was also an *example* of a convergent sequence and subsequence. Associated *tools* to the *strategy* were: the *graphical representation*, the *example* of a sequence s_n given by a student, the *example* of a subsequence r_m given by another student, and Zenobia's *questions* to students (such as asking all students to put values of s_n, ε, and k_0 on the graph).

Conclusions

In our analysis of Zenobia's practice, we reported on her teaching *strategy getting intuition by visual reference/reference to examples*. This report could add value to lecturers' potential lack of strategies in teaching proof (Weber, 2010). Zenobia's *strategy* contributed to a small group culture utilising her research strategies, such as "understand what everything means" and "get some intuition". So, the students potentially made sense of the meaning of notation in the definitions and got intuition for the definition of convergence. Our interpretation of the roles of proof for this *strategy* is students' gaining insight into why the statement of the proof task is true (*explanation*); and discovering results new to students (*discovery*). It seems that Zenobia's goal of students' experiencing of 'what it feels to understand mathematics' is related to experiencing these two roles of proof. It also seems that Zenobia's *strategy* revealed *discovery* as a role of proof which Yopp's (2011) lecturers did not self-report.

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Pedagogical mathematics for student exploration of threshold concepts

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Pedagogical Mathematics refers to mathematical questions or problems which arise in the context of pedagogical issues. Far from the boundaries of mathematical research, they nevertheless provide stimulus for mathematical investigation. Many are of use as initial motivation for a topic, for getting to grips with a topic, and for revising, reviewing and deepening appreciation and comprehension of a topic. Examples are given from the domain of linear algebra, arising from the creation of an applet to designed to display the geometric basis of matrices, bases, and eigenvectors.

Introduction

University mathematics abounds with *threshold concepts*: concepts that need to be deeply comprehended, appreciated and integrated into learners' functioning so as to make progress and which, once appreciated, are ever-present to inform related concepts and the carrying out of procedures (Meyer & Land 2003). Sometimes they even obscure the passage or route taken to reach them, so deeply embedded do they become. Examples include functions, equivalence relations, differentiation and integration, … . In mathematics there are many different concepts which could be considered core or threshold, and yet what for one course is a threshold concept may be peripheral for another. However, identifying threshold concepts for a particular course of study and arranging for students to encounter them appropriately is pedagogically vital in caring for students' learning. Identifying such concepts is one thing; working on developing a rich concept image (Tall & Vinner 1981) to accompany formal definitions of them is another; internalising them is yet another.

In addition to threshold concepts there are ubiquitous themes that pervade mathematics, such as *doing and undoing*, which includes i*nverses* and *conjugation*, what Melzak (1983) called bypasses; *invariance in the midst of change*; *freedom and constraint*; and *organising and characterising* (Mason 2011; see also Gardiner 1987). These are closely related to *mathematical habits of mind* (Cuoco & Mark 1996).

I offer the conjecture that time spent internalising core concepts and ubiquitous themes is time well spent because it makes further development much more efficient: less effort is required for learning and for teaching. They form the foundations around which other concepts, methods, procedures and concepts accumulate. It therefore behoves lecturers to select the core threshold concepts for their course and to spend sufficient time on these with students, in a variety of ways, so that students really do cross the threshold. These concepts can then provide the basis for reviewing a topic and for demonstrating appreciation and comprehension of concepts.

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Furthermore, internalising, or as Gattegno (1970) put it, *integration through subordination of attention* is much more efficient and effective than mindless rehearsing of routine exercises. It is when uncertainty after uncertainty pile up one upon another that learners lose the plot, feel their confidence draining away, and experience their desire to understand mathematics more deeply drain away.

To this end I shall pose some mathematical questions arising from pedagogical considerations of core mathematical ideas from linear transformations. Mathematical problems that arise from pedagogical issues based on experience of teaching a topic give rise to what I call *pedagogical mathematics*. They often arise for me from making interactive applets for displaying complex mathematical relationships, triggering engagement with significant mathematical questions, often ones which might not otherwise have come to mind.

A further aspect of pedagogical mathematics involves the teacher having recent personal experience of the use of their own mathematical powers, and encounters with pervasive mathematical themes, which parallel the experience learners may have when trying to get to grips with new concepts. This is how to sensitise yourself to the confusions and experiences of learners, enabling you to devise pedagogic strategies and tactics (Mason 2002) to ease learners over 'rough ground'.

Background Theories

Integration Through Subordination

Gattegno (1970) offered the challenging slogan, "Only awareness is educable". Slogans like this can act as a protasis, which when combined with recent past experience, can invoke some sort of syllogistic conclusion (Mason 1998). Gattegno used *awareness* to mean 'that which enables action', and this covers both consciously and unconsciously invoked actions. Using an image of the human psyche as a chariot taken from various sources (Gurdjieff 1950), I extended the slogan to include "Only behaviour is trainable" and "Only emotion is harnessible". The idea is that energy flows from and through evoked emotions, and is available to be harnessed to intentional actions, but otherwise leaks through habitual actions. To educate your awareness is to integrate useful actions which can be enacted in the future, and to sensitise your attention to notice opportunities in the future. Gattegno (1970) observed that integrating actions and attention-sensitivities is done most efficiently by subordinating attention. By this he meant that since expert behaviour appears to be effortless, and does not absorb full attention to the action, the best way to integrate an action is to withdraw attention from performing it. This in turn is best achieved by students working on a task which calls upon them to enact the desired action while their attention is elsewhere. The film *Karate Kid* illustrates this beautifully in the context of martial arts, where in order to loosen the wrist, the boy is called upon to sand and then paint a picket fence: attention is directed to sanding and painting, and away from creating the wrist action. Pedagogic Mathematics used as exploration can achieve integration through subordination of attention.

Concept Images, Example Spaces & Question Spaces

Tall and Vinner (1981) drew attention to the possibility of a gap between students' sense of a concept (their concept image), and the formal definition. The concept image includes associations, familiar examples, mental images, inner incantations when carrying out procedures, and ways of describing the concept in ordinary language rather than in the formal language. Shifting from familiar language to formal language so that the formal language is internalised and the language of choice when expressing their thinking takes time, certainly at the beginning of a university course in mathematics.

A student's concept image includes not simply a few examples that they may have used to gain familiarity, but the whole space of examples (Watson & Mason 2002, 2005) together with tools for tinkering with examples so as to construct new ones. From the point of view of variation theory (Marton 2015), to comprehend a concept is to be aware of what can be varied, over what range, and what cannot be varied or what constraints apply in various examples while remaining an example of the concept.

Concept images include the various associations which may be triggered metonymically, with affective content, as well as metaphorically, with cognitive structure. They also include the actions that become available through these associations, which may or may not be conscious.

It is not surprising therefore that a concept image may dominate any formal definition, at least until students have become familiar with using formal language to express themselves.

The Role of Mathematical Tasks

Students have been given mathematical tasks (problems to solve, exercises to complete) at least since the earliest records of mathematical activity. What are they for?

Tasks can be used to prepare learners for a topic; to introduce a topic; to develop and enrich learner example spaces in the midst of a topic; to review a topic; and to evaluate learner appreciation and comprehension (their grasp or understanding) of a topic. For example, Bob Burn (2008) would often set for homework an impossible task such as inviting students to construct a continuous function on a closed interval of R which does not attain its extremal values; in the next class he would then prove the theorem, knowing that students had some investment in finding out why they had failed. Guy Brousseau (2006) made a rough drawing of the medians of a triangle intersecting pairwise to form a triangle, and then asked students to draw a triangle for which the intersection triangle was big enough to see clearly, ostensibly as a setting for further reasoning, but actually as a means of getting them to realise (literally), conjecture and prove that the medians in fact intersect in a single point.

The conjecture being put forward here is that selecting a few core threshold concepts and retuning to them throughout a course could prompt learners to get to grips with those core ideas, and to enrich both their concept images and their example spaces through repeated exposure to the same ideas.

Assenting and Asserting

Students who sit 'at the back of the class' (literally or figuratively) and *assent* to what is said and done are reinforced in their passive mode by the act of taking notes and attempting to keep up with the lecturer. But in order to learn to think mathematically it is essential to make conjectures and then to test those conjectures. This is the essence of Pólya's film *Let us Teach Guessing* (1965), and the backbone of *Thinking Mathematically* (Mason, Burton & Stacey 1982/2010). It is important that teachers generate a conjecturing or mathematical ethos in the classroom, in which students asserting something, then disbelieve their conjecture and with the help of colleagues seek counter-examples by means of which to modify and improve it.

Extending and Varying

No task is an island, complete unto itself. It is (almost always) possible to extend and vary any result. As soon as a student can solve a particular problem, it is vital that they then seek the class of problems which succumb to the same method by changing one or more parameters. This is the *what-if-not?* questioning advocated by Brown & Walter (1983). Also available is to consider inverse problems in which (some of) what was data becomes unknown, and some of what had to be found becomes data. This *what-if* and *what-if-not* stance towards problems generates similarity classes of problems and crystallises experience.

See-Experience-Master, Manipulating-Getting-a-sense of-Articulating and Enactive-Iconic-Symbolic

Bruner (1966) distinguished three modes of (re)presentation, which can usefully be thought of as three different worlds of experience. *Enactive* refers to actually doing things, manipulating something reasonably familiar (which may be a material object but can also be a diagram or familiar symbols). Pólya (1962) used the term specialising: trying specific instances. The purpose of the manipulating is to *get-a-sense-of* what is or might be going on, seeking relationships which may later be seen as instantiations of properties (as part of generalisation). Often it helps top try to capture those relationships in a diagram (iconic mode). The act of conjecturing a generalisation is an attempt to articulate the underlying structural relationships, and over time these articulations become more succinct and more meaningful. They may begin as long strings of cautious words, but eventually they become sufficiently succinct to be written down, perhaps even in a more formal mathematical expression (symbolic mode). When the symbols become familiar, they act as confidence-inspiring entities for manipulating and specialising in the future. That is why it is worthwhile spending time on the threshold concepts so that student confidence grows and so that they become building blocks of students' appreciation and comprehension, rather than blockages to further understanding.

Explorations in Linear Algebra

The idea is to develop mastery of a topic not through rehearsal of routine exercises but through exploring a question which is challenging, and which calls upon the various technical terms and concepts of the topic. One way of doing this is to prompt learners to extend and enrich their example space; another is to pose challenges that go to the heart of the topic. All of the following arise quite spontaneously when using an applet to display the dynamic nature of linear algebra through varying basis elements and vectors.

For example, having worked on change of basis in vector spaces:

- Given a non-singular linear transformation T from a vector space to itself, what is the set of matrices that can be used to (re)present T?
- Given a linear transformation T from a vector space to itself, which sends the standard basis $\boldsymbol{e}$ to the basis $\boldsymbol{f}$, represented by a matrix M which sends vectors specified in basis $\boldsymbol{e}$ to vectors specified in basis $\boldsymbol{e}$, what is the matrix of T sending vectors specified in basis $\boldsymbol{f}$ to vectors specified in basis $\boldsymbol{f}$?

The answer, to both, when you know, is of course easy. This is characteristic of threshold concepts: when you have crossed the threshold, questions about it are easy, but when you have not, they can seem insuperable. However learners who have only recently encountered the topic of change of basis may require some experimentation or exploration to convince themselves, and being surprised at the generality, may be unsure whether they are correct.

Linear algebra is as much geometric as it is algebraic, its power coming from harnessing the two. There is no need to use numbers when thinking of matrices:

- Any two linearly independent vectors $\{f_1, f_2\}$, will form a basis. The matrix M which transforms the standard basis $\{e_1, e_2\}$ to $\{f_1, f_2\}$ can be presented as an action on any vector v (in terms of the standard basis) which gives an image vector w expressed in terms of $\{f_1, f_2\}$. How might w be constructed geometrically using the new basis $\{f_1, f_2\}$?

To attempt this requires clear thinking as to how the coordinates of a vector are interpreted/constructed geometrically. It could be used before matrices are introduced at all, to offer a geometrical sense of what a linear transformation does, or after matrices have been encountered, to offer a geometrical sense of what matrix multiplication is doing. The applet which was presented at the conference (Mason 2015) offers opportunities to explore these and the following questions. It is entirely number-free (as far as the user is concerned).

A related question is

- How are the basis vectors of the row space of a matrix related geometrically to the basis vectors of the column space? How might this be seen geometrically?

At an intermediate level is the following:

- Linear transformations T from R^2 to R^2 mapping v to w, can be expressed in terms of the images of the standard basis under T, namely $\{f_1, f_2\}$. What is the boundary of the region in which the second vector f_2 must lie, so that T has eigenvectors?

Such a question has always been available to be asked, but when dynamic geometry is being used, it arises as an entirely natural question spurred by curiosity released by the possibility of varying objects. The same question can be asked about f_2 when f_1 is held fixed.

By contrast, the question

- Given a non-singular linear transformation T from R^2 to R^2, the image of a circle is an ellipse. How are the axes of the ellipse related to the transformation T?

requires deep insight into inner products as the means for calculating lengths, and possibly also change of basis, or else considerable exploration, in order to reach a satisfactory conclusion.

For a more advanced group of students studying ring theory, a question such as the following calls upon reviewing and enriching their concept of vector spaces by making use of newly acquired concepts in a familiar setting.

- The set of matrices mapping a vector space to itself and with v as an eigenvector forms a ring (closed under addition and scalar multiplication). What geometrically are the matrices that are 0-divisors?

Even more challenging is the question

- For a given linear transformation T from R^2 to R^2, for which vectors v with image $T(v)$ is $|T(v) - v|$ greatest and for which is it least? How do they relate to the eigenvectors?
- When is the cosine of the angle between v and $T(v)$ at its greatest and its least?

Less challenging but equally powerful for consolidating appreciation and comprehension of a topic is to get learners to construct their own examples meeting various constraints (Watson & Mason 2005). For example,

- Construct a matrix relative to the standard basis of a linear transformation T from R^2 to R^2 for which the eigenvectors are at an angle of 60° and for which one is twice the length of the other. Show how to convert your example into all other possible examples.

Exploiting Pedagogy-inspired Mathematical Problems

Showing students an animation means that they have seen something flash by. No matter how carefully animations are constructed, it is important that students try to re-construct mentally what they have seen, and then, on the basis of questions arising, have the opportunity to see the animation again, stopping it at critical points so as to check their conjecture or reach an interpretation of something they missed. Mental re-construction is one of many pedagogic devices for supporting students in educating their awareness, as they link actions with situations.

Seeing something go by is not at all the same thing as getting experience. Gaining experience is not sufficient for internalising or integrating something, because, as I have said many times, "one thing we don't seem to learn from experience, is that we don't often learn from experience alone." Something more is required. Although traditionally that 'something' is associated with "practice makes perfect", there is little evidence of the adage actually being generally true. However, "practice through subordinating attention" is much more effective. The framework See-Experience-Master can act as a reminder to provide students with the kinds of experiences that they can work on to gain mastery, rather than simply seeing things go by in a rush.

Through raising a number of mathematical questions, I have illustrated how pedagogically active questions can arise for someone who is alive and awake to their mathematical thinking, seeking to vary and extend results with a problem-posing frame of mind. Even deeper appreciation of a topic is available when you try to display mathematical results in a dynamic fashion. It is unfortunate that students don't have the time to create the applets for themselves, but as a second best, displaying phenomena (dynamic and static) and getting students to try to articulate what they have seen, formulate conjectures, and try to justify those conjectures, contributes to deepening their appreciation and comprehension of the particular topic and so to building their confidence in their grasp of the topic, while at the same time enriching their mathematical experience of mathematical thinking.

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Calculus I teaching: What can we learn from snapshots of lessons from 18 successful institutions?

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As part of the Characteristics of Successful Programs in College Calculus (CSPCC) project we observed nearly 70 lessons taught by 65 instructors at 18 institutions. The observation protocol included a Problem Log with which observers recorded the mathematical tasks used in class and their enactment, specifically who was involved in the solution (e.g., groups, teacher), what representations were called for (e.g., symbolic, graphical), what technology was used (e.g., graphing calculators) and other important features (e.g., doing a proof). In all we analyzed 497 tasks. In this presentation we discuss what we learned about calculus teaching by analyzing the tasks used and their enactment and the affordances of the instrument used to collect classroom data.

A first-year calculus course in the United States (Calculus I) provides the basic tools for studying the mathematical concept of *change*. In contrast to other counties, calculus in the United States is usually a first year university course and it is typically required for any science, technology, engineering, or mathematics major. In Fall 2010, over 300,000 students nation wide were taking a calculus course at their college or university (Blair, Kirkman, & Maxwell, 2013). A persistent problem of the teaching of university calculus is the perceived high failure rate in the course (Bressoud, Mesa, & Rasmussen, 2015). Students' disengagement is usually a major reason: lectures are uninspiring or unimaginative, the curriculum is "overstuffed" and taught at too fast a pace, and instructors show little concern for student understanding (Seymour & Hewitt, 1997). When students fail the course, their opportunities to pursue STEM fields are curtailed. The large proportion of students failing calculus contributes to the image of calculus as a filter (Steen, 1988). In the late 80s several efforts to change the nature of the curriculum and the teaching of calculus resulted in various changes that have percolated through current day curricula. The "Harvard" calculus, for example, makes heavy use of the rule of four (verbal, tabular, symbolic, and graphical representations), technology (notably graphing calculators and computer algebra systems), and conceptual and contextualized problems (see e.g., Hughes-Hallett, Gleason, McCallum, & Others, 2005). With the goal of identifying institutions that were especially successful in keeping students from dropping out of the calculus sequence, the *Characteristics of Successful Programs in College Calculus* (Bressoud, Rasmussen, Carlson, Mesa, & Pearson, 2010) was launched in 2010. The study, conducted in two phases, generated survey and case study data that provide an unprecedented source of information about Calculus I students, their teachers, and their programs from a cross-section of over 150 post-secondary programs across the United States. In this paper we focus on a slice of the data collected as part of the case study phase of the project, specifically the observations of calculus lessons we conducted in the visits to

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the 18 case study institutions that were selected because they were successful[1], and use these observations to assess the extent to which lessons at these institution exemplify changes proposed by the U.S. calculus reform of the 80s.

Teaching in higher education

Teaching has been an important topic in the higher education literature (Menges & Austin, 2001). Recent theorizations of teaching acknowledge the situated and historical nature of the activity, carried out by individuals, bound by disciplinary expertise and immersed in very specific contexts (e.g., Hora & Ferrare, 2012). The more specific construct, *instruction*, defined as the interactions between the teacher, the students, and the content, within specific environments (Cohen, Raudenbush, & Ball, 2003), has been instrumental in investigations of mathematics teaching at community colleges (e.g., Mesa, Celis, & Lande, 2014). These conceptualizations affirm the importance of studying how the interactions occur in real time in order to account for the ways in which learning can be facilitated. Studies of mathematics classrooms at research universities suggest that core instructional practices include lecturing while working out specific problems, with the goal of imparting information (Hora & Ferrare, 2012) and that there is minimal use of complex problems, use of technology, or group work. These studies suggest that student participation in mathematics lessons is highly guided by the instructor and that there is an emphasis on solving problems at the board with the intention of demonstrating, rather than creating, knowledge. These studies however, lack the specificity of content that a single course provides and they do not look at 'successful' practice. In this study we rely on observation of calculus lessons taught at the successful institutions to characterize the problems teachers and students in class. Specifically we asked: what are the mathematical and pedagogical features of the problems used in Calculus I lessons at institutions with successful calculus programs? This information helps determine the level of alignment between the implementation observed and the ideals of the US calculus reform.

Methods

We observed close to 70 lessons taught by 65 different instructors. Each section was observed only once. Data were collected via an observation instrument designed to capture features of calculus instruction that we knew were present: lecture, interaction between teachers and students, and mathematical tasks (also called, simply, "problems"). For each problem we recorded the start and stop time of the problem, the problem statement, its solution, and circled codes attending to four dimensions: who performed the problem (Actor: the lecturer, the class, individual students, students at the board, students in pairs or students in groups), what technology was used in the problem (scientific calculator; graphing calculators; computer algebra systems; animations), what representations were used (graphical, tabular, symbolic, verbal) and other complexity features (it required Proofs/Justifications, solution focused on Skills/Manipulation, the problem was Open Ended,

[1] Success was defined by a combination of student variables from the survey (e.g., persistence in calculus as marked by stated intention to take Calculus II and affective changes, including enjoyment of math, confidence in mathematical ability, and interest to continue studying math) and program passing rates (Hsu, Mesa, & The Calculus Case Collective, 2014).

used a Diagram, was Contextualized, Multiple Methods were presented or discussed, see Figure 1). We analyzed the data at the two levels, problem and lesson. Descriptive summary statistics of data at the problem level give us a picture of what this corpus of problems looks like that we use to get a sense of what students might be experiencing in a given moment in time as they take a calculus lesson. There are limitations due to the small number of observations per teacher and the selectiveness of the institutions in the sample. The findings are not meant to be generalizations but rather descriptions of the phenomenon observed at these institutions as a whole.

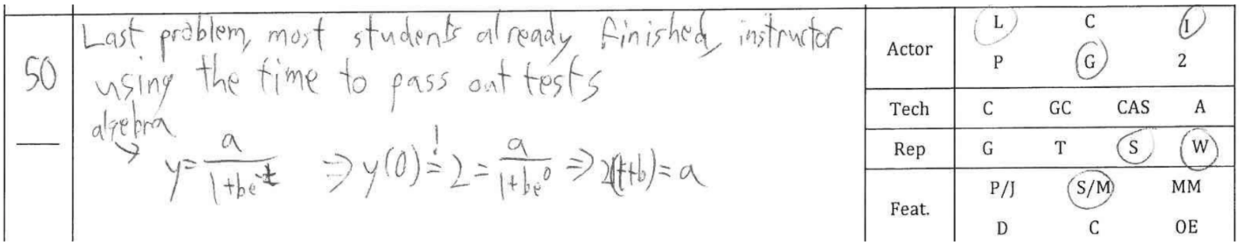

Figure 1: Problem log entry (White, Blum, & Mesa, 2013).

Findings

We recorded 497 problems across 67 lessons. On average the time spent on problems was high (~10 minutes), but the distribution of time was skewed: 23% of the problems took between 1 and 3 minutes, with the median being 6 minutes. Ten percent of the problems (45) took 20 minutes or more to complete. Instructors were in charge of presenting a large proportion problems (82%); other forms of engagement—group, pair, or class—were less common (each present in less than 3% of the problems), although students were observed working individually on nearly 15% of the problems. Technology use was also not very common, being observed in about 3% of the problems. Symbolic representations were used in almost two thirds of the problems, whereas graphs were present in almost a fourth of the problems. Finally, almost two thirds of the problems had a solution that emphasized skill development, whereas just 12% of problems were contextualized, and 9% required a proof or a justification. A small 3% and 2% of problems were open ended or solved in more than one way, respectively.

Discussion and Conclusion

The analysis of the problems in these Calculus I lessons from institutions with successful calculus programs suggest an astonishing homogeneity of practices. The analysis of the problems in these lessons suggest that: 1. for the most part the instructor is in charge of presenting problems, 2. the majority of the problems seek the reinforcement of skills and methods using mainly symbolic representations, and 3. technology, and other student centered class organizations were less frequently used. Two explanations can justify these findings, content and difficulty of change. A non-negligible number of lessons dealt with topics for which reformed features may not have been helpful in achieving the lesson's goal (e.g., learning to use the L'Hôpital's rule). Thus a more nuanced analysis needs to account for specific topic of each lesson to determine whether content played a significant role in these findings (e.g., curve fitting may rely on technology). We also need to investigate whether the opposite is true, that is, that there were a number of topics for which more interesting task features (e.g., pairs, graphing calculators, justification/proofs, multiple representations) could have been possible, but were not observed. Second, the literature indicates that changing educational practices is difficult (Dancy & Henderson, 2009). Perhaps the changes proposed by the calculus reform required difficult-to-secure resources (e.g., smaller class sizes instead of large lecture halls) or encountered faculty or student resistance, all of which can dampen efforts to make changes. Our analysis of the CSPCC survey data shows that student-centered instruction (group work, word problems, "flipped" class, student explanations of thinking) had a small and negative impact on students' attitudes towards calculus (Sonnert & Sadler, 2015). The connection is not simple to tease out with the current data, however, especially because the observations occurred in "successful" institutions. Our sample is small, but we believe that the observation process allowed us to gather information that helps characterize the Calculus I lessons that over 2000 students were experiencing at a very particular moment in time. Our current ongoing analysis indicates that the instrument is useful in capturing the features that help distinguish various types of enactments. This type of data also allows for clustering techniques that can be used to describe types of lessons, thus helping to describe students' learning opportunities in this key, gateway course to STEM majors.

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Why different mathematics instructors teach students different lessons about mathematics in lectures

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Lectures in courses for math majors typically revolve around definitions, theorems, and proofs, but the lessons students learn about mathematics extend far beyond the scope of the content discussed explicitly. Still, there has been little empirical research in this area. In my research, I explore what mathematical ideas different instructors try to convey in lectures and why. By comparing lectures in different sections of the same courses, and through interviews with the instructors aimed at eliciting their beliefs and goals, I have found that different practices for preparing for lectures lead instructors to different student profiles and consequently have great impact on the mathematics in the lectures. This finding has several potential implications for professional development at the collegiate level.

Introduction

What do we want students to learn in advanced proof-oriented math courses at university? What ideas and lessons about mathematics should instructors teach in lectures and how?

There is a widely held criticism that lectures at university comprise almost entirely of cycles of polished formal expositions of definitions, theorems and proofs, presented in solemn and unrelieved concatenation (Davis & Hersh, 1998; Dreyfus, 1991). However, the perception that the mathematical content in lectures comprises only of definitions, theorems and proofs can be contested on at least two grounds: (1) There is a substantial body of literature (e.g. Schoenfeld, 1988; Yackel & Cobb, 1996) describing how students pick up practices, ways of thinking and perspectives about mathematics through instruction regardless of whether these ideas are discussed explicitly in classrooms or not; (2) Recent empirical studies that explored lectures in advanced math courses from the perspective of the instructors suggested that the mathematical ideas instructors try to convey extends above and beyond the content that is written on the board or discussed explicitly in the classrooms (e.g. Lew et al., in press).

This study explores the connections between instructors' practices and beliefs, and the consequent impact on the mathematics addressed in the lectures, by observing lessons and through interviews with different instructors teaching different sections in the same courses. The goals of the study are: (1) To explore the mathematical ideas and lessons about mathematics that the different instructors try to convey in the lectures; and (2) To propose explanations as to why instructors address the mathematics in their lectures the ways they do.

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Theoretical Background

In terms of research and innovation, mathematics at the university level lags many steps behind K-12, as the curriculum and teaching practices in most university classrooms is essentially the same as it has been for decades, and most advanced math courses are typically taught in a lecture format. Math lectures have been criticized repeatedly over the years by mathematicians and math-educators for communicating mathematics in a finished and polish form, usually following the sequence theorem-proof-applications (Dreyfus, 1991), and for depriving students the opportunity to experience and learn from the processes by which new mathematical ideas are generated (Davis & Hersh, 1998). However, these views of lectures are generally based on personal experiences and shared opinions (Lew et al., in press), as empirical research on the actual teaching practices at the university level is virtually nonexistent, and very little is known about what university math instructors think and do on a daily basis as they perform their teaching work (Speer, Smith, & Horvath, 2010). In recent years, several case studies of instructors at advanced mathematical courses have portrayed a significantly different profile of lectures than the one described by critics (e.g. Lew et al., in press; Pinto, 2013, in press).While these studies challenge the general perception of lectures and can lead to a better understanding of instructors' teaching approaches and methods, the relationships between what mathematics instructors try to convey and how, and the ideas and lessons about mathematics that he students learn, remains mostly an uncharted research area.

In this study I use Schoenfeld's resources-orientations-goals (ROG) model for decision-making processes (2011). The ROG model was developed as a tool for explaining how and why teachers make their instructional decisions they make in terms of knowledge, orientations (e.g. beliefs, views) and goals. Substantial empirical data has been subsumed under the ROG umbrella (e.g. Pinto, 2013, in press) and it has proven its usefulness in uncovering connections between particular beliefs and specific practices, inside and outside the classroom.

Methods

Data were collected at two large public universities, one in Israel and the other in the United States, from 7 different sections in two Real Analysis courses. The instructors in this study were two graduate-student instructors (GSIs), two math lecturers, and three mathematician-instructors. The GSIs based their lectures on the same lesson-plans while the other instructors followed, in varying ways and degrees, their course's book. The author attended the lectures throughout the semester, taped them and took notes. The lectures were compared to the curriculum to highlight potential instances where the instructors tried to convey content beyond what was specified in the curriculum. These instances served as the focal points of discussions with the instructors that aimed at eliciting the mathematical ideas the instructors tried to convey, and the considerations underlying their instruction. The interviews were transcribed and analyzed according to the ROG framework to identify connections, first between the instructors' beliefs and practices, and then to the mathematics in the lectures.

Findings

The instructors teaching different sections in the same course relied on the same curriculum. Yet, in large measure, because of what they considered to be important about the content, what emerged in class, and what the students experienced, was radically different. The reflections of the instructors on their adaptations revealed that in many cases the instructors addressed the content in a way they hoped would convey important mathematical ideas. A discussion of the adaptations the instructors made and the mathematical ideas they were trying to convey is beyond the scope of this paper and will be presented separately. However, there are two important points to note. First, we note that instructors were conveying ideas beyond what was specified in the curriculum in all parts of the lectures. The few examples in the literature that the author is aware of that studied the content in the lecture from the perspective of the instructors have focused on the presentations of proofs. However, the instructors participating in this study have tried conveying valuable lessons about mathematics while motivating theorems, discussing examples and special cases, solving exercises, and introducing new concepts as definitions. The second point to note is that when considering the mathematical ideas the instructors had in mind and tried addressing in the lectures as a whole, it turns out that a significant portion of the content in the lecture remained implicit, that is, it was left for the students to infer, without being framed as a content that is taught or should be learned. Furthermore, in most cases these ideas were discussed orally without being recorded on the board, which according to the findings in (Lew et al., in press) suggests that students may have not recognized these ideas as a something they can and should learn.

Another finding of this study is that most adaptations the instructors made were not the result of in-the-moment decisions, but rather a consequence of conscious and deliberate decisions made prior to the lecture. This observation highlights a significant difference between instruction in the collegiate level and instruction in K-12, where the course of a lesson is often determined by interactions between the teacher and the students, and the consequent in-the-moment decisions the teacher makes. Instead, the nature of a lecture at university, where the instructors do most if not all of the talking, leads to a much bigger role of the lesson image – the instructors' full envisioning, before the lectures, of how the lecture will play out in practice. Thus, the practices of instructors for preparing for a lecture, and the factors that shape these practices have a crucial impact on the mathematics in the lectures.

Recurring patterns in the reflections of the instructors on their pedagogy, suggested four major categories that shape the lesson-image: The student profile(s) the instructors have in mind (student image), the model the instructors have of a good instructor (instructor image), the sense of the mathematical experience the instructors want their students to have, as well long-term and short-term learning goals and outcomes (content image), and finally the constraints and expectations of the institution. These four categories were evident, to different extents, in almost every reflection of the instructors on their pedagogy. Two important observations to note are that by and large these images were independent of the specific content and the actual students in the classrooms, and that these images seemed to remain almost fixed in the discussions throughout the semester. This phenomenon may be due to the fact that instructors got very little feedback from their students during the semester and had little opportunities to challenge and refine their perceptions and understanding about how their student are doing, and what works in the lecture.

One notable approach instructors have for preparing for lectures is to interact with the mathematics – make sense of the content, solve the exercises and prove the theorems on their own, and then reflect on what they were doing and use this reflections as a resource and a source of inspiration for discussions in the lectures. An example of instructor using this practice, and how it shaped the lecture, can be found in (Pinto, 2013, in press). Another notable approach for preparing for lectures is to try and scan the curriculum from the perspective of the students, and try to identify potential learning obstacles, misconceptions and difficulties students may face, as well as connections to students prior knowledge. This practice relies on, and is shaped by, what the instructors knows or believes about his students. As it turns out, the instructors in this study tended to rely on just one of these approaches. The two GSIs and two of the mathematicians relied mostly on self-reflections, while the third mathematician and the two experienced lecturer relied on different models of students.

There was a strong correlation between the practices the instructors use and their student image. Not surprisingly, in the case of the instructors that relied on self-reflection, the student image had many characteristics in common with the instructors themselves. Consequently, in many ways, the instructors were teaching future math researchers, and in their lectures they put great emphasize on modeling their own practices and ways of thinking. In contrast, the student image of the instructors that relied on different models of students was far less homogenous, and more independent of the instructors. These instructors were less prone to model their own behavior and ways of thinking, and instead modeled an approach that was closer to the students, for example by writing mathematics on the board in the same way as expected from students. The instructors also placed greater emphasis on misconceptions, as well as procedures and rituals, for example while reading definitions, or starting a proof.

The findings discussed in this paper have several implications for professional development at the collegiate level. All the instructors in this study were highly motivated and their reflection on their teaching indicated high pedagogical awareness. Furthermore, the instructors were clearly acting in the best interests of their students, according to their own perspectives and understandings. However, limited understanding of the students sometimes constrained the instructors, as one of the GSI noted: "*I know it is naïve to think that I and the students would find the same things interesting or confusing, but the way I see it is an inevitable working assumption.*" Thus, this study highlights the importance of addressing preparation for a lecture, and helping instructors develop practices that would make them less dependent on self-reflections, and lead to instruction that fits better the actual students in the classrooms.

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An insight into mathematics tutorials

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This presentation reports findings from university mathematics teaching in the tutorial setting. Although the tutorial, in addition to the lecture, plays a vital role in university mathematics education, there is hardly any research on teaching assistants in Germany. To provide as much insight into their work as possible, teaching episodes are analyzed from various mathematical and didactical perspectives. Among others, the constructions of mathematics in tutor-student interactions are discussed. In this paper, the focus is on ways the teaching assistants use to discuss exercises. We call them "scripts". The five scripts that could be identified in 78 discussions are presented and illustrated by examples.

Introduction

Much research has been conducted on how and how well teachers teach mathematics in school. In the last decade, educational studies became more and more concerned with tertiary teaching, however, the focus is mostly on lecturers and students. Little is known about the work of teaching assistants (TAs), who play an important role in mathematics university teaching. Teaching assistants in mathematics in Germany are usually undergraduate students from mathematics teacher education programs, who teach small group tutorials with up to 30 students and often function as a link between lecturers and students. Many universities have recognized the impact TAs can have on the students' learning and have developed special trainings, but only few research has been done on how they teach and how they could be supported more effectively (e.g. Mali, Biza, & Jaworski, 2015).

The LIMA project developed a training program (e.g. Biehler, Hochmuth, Klemm, Schreiber, & Hänze, 2012) orientated on the specific needs of mathematical teaching. In order to identify the needs of our teaching assistants, we made a theoretical competence analysis and observed teaching assistants during their work in the tutorials. However, a lot of input for the training was generated from experience and pedagogical literature. The question whether these underlying assumptions could be confirmed by research remained. This is where the author's doctoral project focuses on: its aim is it to point out the challenges for teaching assistants and to gain inputs for training.

The doctoral project consists of three major studies. The first one is a theoretical discussion, comparing teachers and teaching assistants in order to determine to which extend results of educational studies related to school (e.g. Helmke, 2012) can be transferred or adapted to tutorials. The second part is based on studies of educational scripts (e.g. Pauli & Reusser, 2003; Seidel, 2002), trying to identify typical scripts of mathematical tutorials. It gives an overview on the work of the TAs and is followed by a more detailed case study analysis of five discussions on one selected exercise. This third part examines tutorials, especially tutor-student interactions, from mathematical and didactical perspectives.

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Theoretical Background

Schank and Abelson were the first to define scripts in 1977 and described them as „a predetermined, stereotyped sequence of actions that defines a well-known situation" (Schank & Abelson, 1977, p. 41). This definition can be used in various contexts, for instance, Schank and Abelson use a restaurant script for illustration. Scripts have also been used in educational research, often to describe education in a non-laboratory setting. By using educational scripts, focusing on teaching activities, it is possible to reduce the high complexity of education without having to use a laboratory setting (e.g. Blömeke, Eichler, & Müller, 2003). Pauli and Reusser (2003) conducted one of the greatest studies on scripts in mathematical education, comparing German and Swiss data from TIMS-Study of the years 1995 and 1999. By analyzing the learning activities the teachers use in their math lessons, Pauli und Reusser were able to identify two main scripts for discussing new topics. The script dominating German lessons (78%) was rather teacher-centered. This script was also very present in Swiss lessons (58%), however, 30% of the Swiss lessons had a more student-centered script (Geman lessons: 13%). Pauli and Reusser also found out that opening activities are present in about half of the lessons, the two most prominent activities were the "correction of homework" and "repetition of prior knowledge". Closing activities at the end of the lesson were used in less than 10% of the lessons. For the main part of the lesson, Pauli and Reusser also identified what methods the teachers used to introduce a new topic, how they motivated new topics and whether they gave time for exercises.

Design of the Study

The study is being conducted at the University of Paderborn, where preservice teachers of mathematics are expected to attend lectures and small group tutorials of 10 to 30 students. Tutorials are 90 minutes weekly sessions. Part of the time is used to discuss exercises students had to work on beforehand. The teaching assistants have corrected the student's work and therefore know where the students struggled. Teaching assistants and lecturers meet weekly in order to discuss the students' difficulties and to plan the tutorial. Sometimes, the teaching assistants get a lot of directions on what they have to discuss and what kind of methods they have to use, but in most cases they are rather free to plan their tutorial as they want to. The teaching assistant gets a model solution from the lecturer which is not made available to the students.

The videos for this study were generated in tutorial trainings over several semesters, they were used to give feedback to their work as teaching assistants. The topics of the tutorials range from analysis to arithmetic and didactics of geometry. Now, this data, containing 78 exercise discussions in 32 different tutorials, is analyzed in more detail. Most of the tutorials (24 of 32) were led by two teaching assistants, however, in most cases only one TA was in charge during the discussion.

The qualitative content analysis (QCA) is used to analyze this large amount of video data. The thematic QCA by Kuckartz (2012)[1] seems to fit the research aim best, as it allows inductive coding and also reduces the big amount of data. The discussion of exercises is divided

[1] for a precise description of the thematic QCA see Kuckartz (2012, p. 77ff)

into three main phases: start up, phase of discussion, finishing up. The first and the last phase are coded for learning activities according to the study of Pauli and Reusser (2003). A learning activity is a well-defined situation in which the teaching assistant or a student takes over a special activity like "clarification of the task difficulty" or "summarizing the results".

The discussion phases are analyzed for three different main categories: methods, completeness and didactical elements. The subcategories are generated by subsumption (see Schreier, 2012, p. 115f.), however, results from research on characteristics of classroom teaching (e.g. Brophy, 2000; Helmke, 2012) influenced the construction of categories. For example, "clarification of expectations" (e.g. Brophy, 2000, p. 31) is often mentioned as a characteristic of good teaching and can be found as a subcategory in the main category "didactical elements". The following subcategories were constructed in this process:

phase	**categories**
start up	opening activities • clarification of task • orientation for process of discussion • feedback on work of students • repetition of relevant topics • clarification of task difficulty
discussion phase	completeness • yes • no methods • discourse • presentation of TA • student presentation • work in groups • individual work didactical elements • visualization • highlighting common mistakes • reference to lecture, other exercises, school • draft solution • clarification of expectations • solving in several ways • recapitulation • clarification of student questions • generating cognitive conflicts • return of student questions • advanced questions
finishing up	closing activities • clarification of questions • summary of exercise

	• advising students to do something • continuation of contents

Table 1: Categories in the three phases identified by inductive coding

Using these categories, each of the 78 discussions is assigned one of five different scripts.

Results

The following list describes these five scripts that could be reconstructed.

- *presentation of the model solution without didactical reflection*

  The TA presents his solution without using many learning activities or didactical elements. The solutions are usually complete, except for small parts of the exercise, and quite similar to the model solution of the exercise.

- *presentation of the model solution with didactical reflection*

  The TA presents his solution while using some learning activities or didactical elements, e.g. "giving feedback on the work of the students". The solution is usually complete, except for small parts of the exercise, and quite similar to the model solution of the exercise.

- *discussion of selected difficulties*

  The TA presents only parts of the solution while using some learning activities or didactical elements. He often points to and explains specific mathematical problems the students had in their work. The presented solution is usually incomplete and differs from the model solution.

- *conveying of strategies to solve a specific type of exercises*

  The TA can present the whole solution or only parts of it. He uses specific learning activities or didactical elements, e.g. "clarification of the task difficulty", and hints at the steps the students have to take in order to solve this type of exercises. The focus of discussion is not on mathematical difficulties, but rather on the procedure.

- *clarification and implementation of a mathematical concept*

  The TA can present the whole solution or only parts of it. He uses specific learning activities like "referring to lecture" and spends some time on explaining a specific mathematical concept.

The following examples are to illustrate the scripts above and to clarify in which way they differ. Therefore, one exercise which is discussed by the teaching assistants Andrew, Oscar, and David is analyzed in more detail. The exercise is the following:

Exercise

Verify, using definition 3.1.3, that the following sequence $(a_n)_{n\in\mathbb{N}}$ converges.

$$a_n = \frac{1}{\sqrt{n}}$$

The lecturer presented Definition 3.1.3 two weeks before:

> **Definition 3.1.3**
> A sequence (a_n) converges to a real number a if, for every positive number ε, there exists an $n_\varepsilon \in \mathbb{N}$ such that whenever $n > n_\varepsilon$, it follows that $|a_n - a| < \varepsilon$.

Andrew uses the most frequent script, the "presentation of the model solution without didactical reflection", to discuss this proof. He starts out with a clarification of the task, then discusses the proof in instructive discourse for 10 minutes and ends up with encouraging the students to ask questions. The following table shows how his discussion can be divided into the three phases:

Phase	**presentation of the model solution without didactical reflection**
start up	opening activities: clarification of task
phase of discussion	method: instructive discourse completeness: yes didactical elements: none
finishing up	closing activities: opportunity to ask questions

Table 2: Script for "presentation of the model solution without didactical reflection" on the example of Andrew

Andrew's discussion is very typical for this script: he presents the solution in some kind of discourse without pointing out specific difficulties, presenting alternative solutions or using other kinds of didactical elements to especially enhance the learning process of the students. The activities of the students are often limited to listening to the TA, taking notes and answering questions.

Discussions with didactical reflection often show different opening activities and more didactical elements. Oscar, for example, who discusses the same exercise as Andrew, highlights one major student difficulty. In the proof, the students had to argue why the following inequalities are equivalent:

$$\left|\frac{1}{\sqrt{n}}\right| < \varepsilon \Leftrightarrow \frac{1}{\sqrt{n}} < \varepsilon$$

Oscar has corrected the students' proofs and is aware of this common difficulty. He makes the students discuss this problem:

21	Oscar	*[...] The absolute value was missing from one step to the next without any kind of reasoning. But you have to argue here. You have to write somewhere why you are not using them anymore. Yes. How can we argue here to omit the absolute value? One alternative? (2 sec) Harry?*

In order to make such kind of comment, he has to identify difficulties students had in their proofs. However, as he still presents the whole solution of the exercise, this discussion is coded as “presentation of the model solution with didactical reflection” and not as “discussion of selected difficulties”.

Phase	**presentation of the model solution with didactical reflection**
start up	opening activities: clarification of task
phase of discussion	method: instructive discourse completeness: yes didactical elements: • highlighting common mistakes • reference to lecture • draft solution • clarification of expectations • solving in several ways
finishing up	closing activities: none

Table 3: Script for “presentation of the model solution without didactical reflection” on the example of Oscar

Table 3 shows that Oscar uses more didactical elements to support the students. Whether he planned all these learning activities before the tutorial or whether he just reacts in the situation cannot be determined.

David uses a third script for the same exercise. The beginning of his discussion indicates that his objective is not only to discuss difficulties students had to prove the convergence of the sequence, but to convey strategies of how to solve such kind of exercises:

1	David	*So, in exercise 2 you had to verify that the sequence converges by using the definition. The sequence is (writes on board):* $a_n = \frac{1}{\sqrt{n}}$. *And you were supposed to use Definition 3.1.3. To begin with, can anyone recall Definition 3.1.3., just the content, not word for word? (2 sec)*
2	David	*We could try it together, if / (writes on board). (10 sec)*
3	David	*What’s the key aspect of Definition 3.1.3? (7 sec)*
4	David	*No one? You know at least a part of it. It doesn’t have to be perfect.*

In the above episode, David tries to call into memory the definition by posing many questions. The underlying strategy David might try to convey is to collect every information that the exercise provides and find out what helps you to solve it. This strategy can be used for many exercises. In the following discussion, it becomes obvious that David especially wants to enable the students to handle this type of exercises. He recalls the different steps they have to take: state what you have to prove, assume a limit and then prove that the assumed limit is correct. His objective becomes even more obvious in some of his following statements, for instance in:

21	David	*[...] Now, we are going to discuss a topic that not everybody has to understand, but you hardly always proceed in a similar way. Okay. It would be great, if you could write that we have to show this (points at definition on the board). [...]*

He focuses on technical skills, understanding the concept of convergence is not part of his objective. As an advanced student, David knows that this technical skills will be more relevant to pass the upcoming exam. He uses didactical elements like “clarification of expectations” which are very common for this kind of script.

Phase	**conveying of strategies to solve a specific type of exercises**
start up	opening activities: none
phase of discussion	method: instructive discourse completeness: yes didactical elements: • highlighting common mistakes • reference to lecture • draft solution • clarification of expectations • solving in several ways • visualization
finishing up	closing activities: none

Table 4: Script for “conveying of strategies to solve a specific type of exercises” on the example of David

The analysis of these different discussions shows, that TAs use different scripts for the same exercise. Therefore, there have to be other factors than the exercise type which influence the TAs in their choice of scripts.

The coding is still in progress, however, the dominant script for the discussion of exercises is clearly “the presentation of the model solution without any didactical reflection”. As the TAs have corrected the students’ solutions beforehand, it is quite surprising that they still discuss the whole exercise and do not focus on difficulties or strategies. However, it is possible that they only satisfy the students’ needs for a model solution. The lack of didactical reflection can be due to the fact that it requires many competences, like e.g. diagnosis of students’ mistakes, which are even a challenge for experienced teachers.

Conclusions

In this short paper, the reader should gain some insight into what happens in mathematics tutorials. The use of scripts makes it possible to characterize the very complex process of classroom teaching.

In this analysis, five scripts TAs use to discuss an exercise could be identified. These range from the presentation of the model solution to the clarification of mathematical concepts and strategies. The most frequent script is the "presentation of the model solution without didactical reflection". Its dominance is somewhat surprising as the teaching assistants know the students' difficulties from the correction and could therefore concentrate on them in the discussion. The reasons for the preference of these scripts can only be assumed.

As some examples illustrated, TAs use different scripts even for the same exercise. Therefore, other factors than the exercise seem to influence their use of scripts. One important factor might be the students: their performance in solving the exercise beforehand and their contributions in the discussion might lead the TAs to change their intended script. In addition, every TA has a specific type of explaining and identifies with his role differently. Also external factors like demands of the teaching team or time restrictions can influence the type of discussion. These different factors could be discussed in tutorial trainings so that the teaching assistants become aware of the different scripts and consider all of them when planning their tutorials.

Further studies will give more insight into the characteristics of tutorial teaching and regard more perspectives, trying to add some results to the research on teaching assistants. In addition, this doctoral study can hopefully contribute in increasing the attention on quality of education on the tertiary level.

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How lectures in advanced mathematics can be ineffective: Focusing on students' interpretations of the lecture

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In this report, we synthesize studies that we have conducted on how students interpret mathematics lectures. We present a case study in which students in an advanced mathematics lecture did not comprehend the points that their professor intended to convey. We present three accounts for this: students' note-taking strategies, their beliefs about proof, and their understanding of the professor's colloquial mathematics.

Research on how students understand lectures in advanced mathematics is sparse. Although lectures are the usual way in which advanced mathematics courses are taught, there are few studies on this practice (Speer, Smith, & Horvarth, 2010). Only recently have researchers systematically investigated how students understand proofs (as opposed to how they check proofs for correctness) and research on how professors choose to present proofs is largely absent (Mejia-Ramos & Inglis, 2009). Our work seeks to address this void in the literature.

In this paper, we first present a case study of one professor presenting a proof in a real analysis lecture (Lew, Fukawa-Connelly, Mejia-Ramos, & Weber, in press). Through interviews with the professor and his students, we noticed that students did not comprehend the main points that the professor was trying to convey. We hypothesized three reasons to account for this communication failure: (i) students' notes focused on what the professor wrote on the board while the professor only stated his main points orally; (ii) students held unproductive beliefs about proof that led them to ignore the main points that the professor was trying to convey; (iii) students did not understand the professor's use of informal *colloquial mathematics*, particularly the use of the term "small" in the context of real analysis.

To further investigate these hypotheses, we first present a large study of lectures and note-taking showing that professors tend to model mathematical behaviors orally and students rarely record these oral comments in their notes. We then present qualitative studies that illustrate how mathematics majors (Weber, 2010) and mathematicians (Lai & Weber, 2014; Weber, 2012) hold conflicting beliefs about the role of proof in lecture and students' responsibilities in reading these proofs. We confirm the generality of these findings with large-scale surveys (Weber, in press; Weber & Mejia-Ramos, 2014). Finally, we illustrate how the failure to understand "small" in a calculus context was also found in a study with a large sample by Oehrtman (2009).

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A case study of a lecture in real analysis

In this case study, we video-recorded a lecture in real analysis given by Dr. A (a pseudonym) at a large state university in the United States, focusing on a ten-minute proof of the following theorem: If a sequence $\{x_n\}$ has the property that there exists a constant r with $0 < r < 1$ such that $|x_n - x_{n-1}| < r^{\,n}$ for any two consecutive terms in the sequence, then $\{x_n\}$ is convergent.

We analyzed this lecture proof in three ways. First, our research team viewed the video and flagged for each instance in which we felt Dr. A was attempting to convey an important mathematical idea to his students. To corroborate our findings, we also showed the same videotape to another course instructor and asked him to make the same judgment. We engaged in this process to see if the main points Dr. A was making were clear and accessible to a mathematically acculturated audience. Next, we interviewed Dr. A and showed him a video of the proof, asking him to stop the recording at each point he was conveying an important mathematical idea. We engaged in this process to see what content was being conveyed from Dr. A's perspective, contrasting it with our own viewing to see if his points were being conveyed clearly. Finally, we interviewed three pairs of students. The details of this interview are described in Lew et al. (in press). For the purposes of this study, we focus on Pass 2 of the interview protocol in which students watched the videotape lecture and were asked what Dr. A was trying to convey in the lecture, and Pass 3 where students were shown the individual clips that (according to Dr. A) contained an important mathematical idea and asked what they thought Dr. A was trying to convey.

The main results were that Dr. A was trying to convey five types of mathematical ideas: (i) Cauchy sequences can be understood as sequences that bunch up, (ii) one can prove a sequence with an unknown limit is converging by showing it is Cauchy (hereon referred to as the Cauchy heuristic), (iii) how one sets up a proof showing a sequence is Cauchy, (iv) the triangle inequality is useful for proving series in absolute value formula are small, and (v) the geometric series formula should be part of one's toolbox to keep some desired quantities small. However, students usually did not cite any of this content as what Dr. A was trying to convey after watching the lecture proof (Pass 2) through our data, even though it was clear to us what Dr. A was trying to convey. In the remainder of this this abstract, we focus on points (ii) and (v), and provide three accounts for why students did not comprehend these ideas.

Students' note-taking in advanced mathematics lectures

There is a large body of research on note-taking in (non-mathematical) lectures. Two important findings are that if students do not record a lecture point in their notes, it is unlikely that they will recall this point at a later time (e.g., Einstein et al., 1985). Second, professors speak at a rate faster than students can write (Kiewra, 1987; Wong, 2014). Hence students cannot be expected to record everything that the lecturer says. They need to prioritize.

In Dr. A's lecture, he spoke of the Cauchy heuristic at three different points. For instance, his lecture contained the following:

> *There's no mention of what the definition is of the sequence, so there's no way we're going to be able to verify the definition limit of a convergent sequence, where we have to produce the limit. So what do we do? […] What kind of sequences do we know converge even if we don't know what their limits are? Cauchy! We'll show it's a Cauchy sequence […] We will show that this sequence converges by showing that it is a Cauchy sequence. A Cauchy sequence is defined without any mention of limit.*

Our research team highlighted this as the main point of the proof and we felt this excerpt conveyed this point clearly. However, no student mentioned this. We noticed that Dr. A only expressed this point orally. He never wrote the Cauchy heuristic, nor any of the other content, on the blackboard. The blackboard only contained the actual proof. When we looked at the students' notes, we found that five students had not recorded any of Dr. A's oral comments. They had only recorded what Dr. A wrote on the blackboard.

To assess the generality of these findings, we studied eight lectures in advanced mathematics and photographed the students' notes after these lectures. We found that the professors commonly modeled productive math behavior but usually did so orally. However, the students' notes rarely contained the professor's oral comments but usually contained comments that the professor wrote down.

Students' beliefs about proof

What students attend to in a lecturer's presentation of a proof is necessarily dependent upon what they think the purpose of a proof is and what their responsibilities are when they read a proof. We investigated this issue by interviewing mathematicians about what it meant to understand a proof (Weber & Mejia-Ramos, 2011), what they were trying to convey when presenting a proof (Lai & Weber, 2014; Lai, Weber, & Mejia-Ramos, 2012), and how students should read a proof (Weber, 2012). Key findings included that in lecture, a proof presentation was about illustrating large ideas and overarching methods; logical details could be found in a textbook. Interviews with mathematics majors revealed that many felt that understanding a proof consisted entirely of understanding how new statements were deduced from previous ones (Weber, 2010).

In a survey with 175 mathematics majors who had completed at least one proof-oriented course and 83 math professors who had taught at least one proof-oriented courses, we found that these results generalized to this larger population. We found that the large majority of the mathematics majors felt that understanding a proof was entirely comprised of knowing how new statements could be deduced from previous ones and that they would not compare the methods that they would take to prove a theorem to the one in the proof that they read. The large majority of mathematics professors thought there was more to understanding a proof than understanding its deductive step-by-step process and they desired that their students compared the methods that they would use to prove a theorem to the method in the proof that they read. (Weber, in press; Weber & Mejia-Ramos, 2014).

Such findings can help account for the results observed in the case study discussed above. For instance, students might ignore Dr. A's description of the Cauchy heuristic because they did not think that proofs should be helpful in expanding their arsenal of proving methods. Also, consider the following excerpt from Dr. A:

> *Dr. A: Now once again we ask the question. If we were to show this is small, we must represent it in terms of what we know is small. Well what do you know is small? For n large enough, the difference between two consecutive terms is small. So what we must do is represent that as a sum of consecutive terms.*

When asked what Dr. A meant by this particular clip, no student mentioned the word "small" (both here and mostly throughout the interview). Rather, they focused on how he set up an equation, saying things such as, "how we can manipulate the problem statement". Hence, students' focus was on manipulation, rather than the proving method.

Students' understanding of colloquial mathematics

In the preceding passage about keeping things small, Dr. A was using what we called *colloquial mathematics*, meaning we interpreted Dr. A as using informal English like "small" to help make technical ideas more accessible to his students. However, we argue that the students did not interpret the word "small" in the way that Dr. A intended. When Dr. A used the term "small", he was referring to quantities being arbitrarily small or sufficiently small (masking the use of universal and existential quantifiers). Dr. A uttered the word "small" eight times in his proof, but students rarely mentioned this in their interviews. When they did, they interpreted small as meaning a short or simplified equation. These results are consistent with a large study with calculus students by Oehrtman (2009). Oehrtman found that despite the calculus professor continually using phrases such as "sufficiently small" or "arbitrarily small", the students rarely used these notions in their reasoning about limits and interpreted these phrases as meaning a very small fixed quantity.

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5. MOTIVATION, BELIEFS AND LEARNING STRATEGIES OF STUDENTS

Beliefs on benefits from learning higher mathematics at university for future secondary school teacher

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This study provides a first insight into beliefs of preservice secondary (Gymnasium, college bound) school teacher about the rationale for learning higher (university) mathematics. The participants of the study were preservice secondary school teachers (n = 31) being at least in their 5th semester. We gave them the task of writing an essay with the topic "Analysis and me". We analyze these essays, in the first part, we inductively extract criteria on what benefits students belief they get from attending university mathematics lectures, in the second part, we compare their statements with normative literature. The analysis shows a wide variety of beliefs concerning potential benefits. However, their views are based on a relatively limited view on teachers' future roles in the classroom.

Introduction

In the last years, teacher education at university has become a subject of research, which gets more and more important. University education of future school teachers consists of courses in mathematics, in mathematics education and in pedagogy and educational science. The future teachers for the Gymnasium attend the same courses of university mathematics as mathematics majors. Other future teachers (primary and lower secondary) usually attend mathematics courses that are particularly designed for this group of students. It is often reported in Germany that many Gymnasium student teachers show motivational problems with university mathematics courses, which they do not consider to be immediately relevant for their future profession. Surprisingly, there is hardly any systematic research on beliefs of these preservice teachers about learning university mathematics. One of the rare studies used a retrospective questionnaire for secondary school teacher (n = 176) after they had finished their studies (Bungartz and Wynands, 1998). The authors found out that the teachers considered the level of mathematics courses at university "too high" and the connection to the intended job "too low". However, the criteria applied by the participants are not explicit in these studies.

Research questions

Our central research questions are about the benefits seen by the student teachers in learning higher mathematic at university.

1. Which benefits do student teachers see in learning higher mathematics at university for their future working as a teacher?
2. Which aspects about their later professional life are articulated by the students and also taken into account in their evaluation of benefits?

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Methodology

To answer the questions above, we used narrative as a research tool. The participants of the study were Gymnasium student teacher who were just starting the course "Didactics of mathematics for grade 10 – 12". This course focused on teaching and learning of calculus. The second author was the lecturer and the first author was the assistant, organizing and giving small-group tutorials. Almost all participants were in the 5^{th} semester, i.e. at the end of the Bachelor of Education (n = 30) and had already attended lectures called "Introduction to mathematical thinking", Calculus 1, Calculus 2, Linear Algebra, "Didactics of geometry", one participant is in the 3^{rd} semester. They were asked to write an essay on the following questions (according to Toerner, 1999):

1. How was your calculus education at school? (Also describe your emotions and attitude)
2. How have your attitudes, your knowledge (e.g. about specific concepts), and your emotions changed because of the calculus lecture at university?
3. What will be the benefit from the calculus lecture for your later professional work as a Gymnasium school teacher?
4. Which new impulse will you pick up out of your experiences with your university education for your own teaching at school?
5. What will you change in your own calculus lecture at school in comparison with your experiences in your school?

Although they were advised to discuss these questions on five pages, most students wrote about three pages. In this paper we concentrate on questions 3 and 4.

Data analysis

Based on Grounded Theory (Strauss & Corbin 1996), we inductively extracted criteria what benefits students express with regard to attending university mathematics lectures and match these with three of four levels of mathematical content knowledge based on Krauss et al. (2013). We distinguish "school mathematical knowledge" comparing to level 2 define by Krauss et al. (2013) as a level of mathematical knowledge required of a good pupil, school mathematics from a higher standpoint" (Klein 2004) comparing to level 3 define by Krauss et al. (2013) which include i.e. a deeper understanding of the contents of the school curriculum and level 4 "University-level knowledge" as the knowledge, which has no overlapping with the content of school curriculum we call "university mathematical knowledge".

Results

In the following, we present what benefits our students see on each level. One benefit mentioned in the essays of attending mathematics lectures at university is the opportunity to practice school mathematics like the calculation of integrals and fill gaps in the mathematical knowledge required at schools. These two points we extracted from the essays were classified as the function of practicing and extending "school mathematical knowledge" at university level.

The level "mathematics from a higher standpoint" (c.f. Klein, 2004) is the most manifold type, although it is not easy to draw a boundary to "university level knowledge". Statements of the students in the essays are pointing out that attending university mathematics courses is important because they learn deeper knowledge of concepts taught in school, for instance precise definitions and proven properties. Mostly, they gave no further explanation, why this is important, but one student hoped to give better explanations using this knowledge to future school students. In addition to this also concepts of proofs in school mathematics were mentioned. There are also statements, in which the students see a benefit in a better understanding of why something is right or why the algorithm learned in school can be used that way. Another point is the establishment of links and connections between separated topics of mathematics. One aim of the students in their future work as a teacher is, that the pupils also see that mathematics is cross-linked. They want to bring school mathematics closer to university mathematics, but for different reasons; some hope that they can get the pupils to see the necessity for proofs, others think that they can motivate their pupils with this and a last aim is to prepare the pupils for the transition to university. For this, a deeper understanding of school mathematics is seen as important. Another benefit from learning mathematics in university is seen in supporting talented pupils and also in answering pupils' questions, meaning not only questions on the curriculum but also beyond it. It was also mentioned in the essays that they could give the pupils advices for their studies, if they know the university courses. They also named other functions of university level mathematical knowledge, which they did not explain in detail: supporting planning lessons, learning options for simplifications and helping to diagnose learning problems of pupils.

Even if there is no direct counterpart of a part of "university mathematical knowledge" in school mathematics, students named benefits of this knowledge for their future teaching. So, university mathematics helps teachers to get a deeper insight into mathematics in order to show pupils the manifoldness of mathematics. Students also see a benefit in their experience that mathematics is not always easy to understand. To struggle with mathematics themselves, might make it easier for them to have empathy with pupils who have learning difficulties in mathematics. Another benefit proposed by the students is that in a few years topics in the school curriculum could change and topics they now only learn in university have to be taught in school. Because of that, they think it is good to learn also topics that have no direct counterpart in school mathematics at the moment. Furthermore, they named a social function as having studied university mathematics will give them authority of an expert mathematician compared to both pupils and other teachers. It was also mentioned that a teacher has to represent the subject at school and because of that it is necessary to have experience with university mathematics. The students also named the development of general mathematics competencies, which will be helpful in solving mathematical problems and challenges at school level and the knowledge of mathematics as a deductive system as a benefit, but without further explanation why they consider the latter as important for teachers.

Comparing these findings with points considered in the literature we can identify some similarities and differences. Shulman says that teachers "need not only understand *that* something is so; teacher must further understand *why* it is so." (1986, p.9). This is also an argument we found in the essays. He pointed out two other dimensions of curricular knowledge: lateral curriculum knowledge “underlies the teacher´s ability to relate the content of a given course or lesson to topics or issues being discussed simultaneously in other classes.” (Shulman, 1986, p.10) this is not mentioned in the essays, but the second dimension: vertical curriculum knowledge which includes “familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school” (Shulman, 1986, p.10) is mentioned by the students when they wrote that they will show in their further teaching that mathematics is cross-linked. The last point is also one of “Mathematical Task of Teaching” listed by Ball et.al (2008). However, Ball et al mention tasks for teachers that are not mentioned by the students like “appraising and adapting the mathematical content of textbooks” or “using mathematical notation and language and critiquing its use”.

Conclusions

By interpreting our data we have to consider the fact that this is no representative sample because of selection effects. First of all, we have to remember that the students who took part in this study are probably “good” students, i.e. these are students who have already passed most courses for their Bachelor degree and studied according to schedule. So we have no essays from students who had big difficulties with their university studies or from students who have even quit their studies. The second point is that not all students attending the course wrote this essay. This selection can also be a positive selection because probably the more committed students wrote this essay. Additionally, the essays were not anonymized and so there is potential for social desirability. Nevertheless, we revealed a wide spectrum of believed benefits of learning higher mathematics at university. It is surprising that most statements point out positive aspects. However, based on a relatively limited view on teachers’ role, students do not mention many aspects that are considered in the normative literature. This may have consequences in redesigning teacher education. Broadening the view of their future roles is important as well as deeper reflecting on the higher mathematics they have learned.

Based on the statements from the essays, we have constructed an interview guide to get a deeper understanding of the beliefs about learning higher mathematics and to specify the knowledge levels. A second aim will be to create a typology of students.

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To defy conventions? – University students' demand of concrete examples and less mathematical formalizations

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Investigating preferences how university teachers like to learn and understand mathematics through lectures and seminars was neglected so far. Thus a study focusing on the mathematical thinking styles of becoming math teachers (N = 219) was conducted. A further goal was, if and how these styles are different dependent if they study for primary, secondary, high or vocational school. One central result of the study was that the students strongly expressed the demand of more concrete examples and less mathematical formalizations within lectures and seminars for a better understanding of mathematics. However precise mathematical notations are a part of the mathematics scientific discipline. How much formalization is necessary? These questions are discussed.

Theoretical background

The theoretical background of the study is mainly based on the theory of mathematical thinking styles developed by Borromeo Ferri (e.g. 2004, 2010, 2015). The presented study is also embedded within the current discussion of teacher education development and higher-education mathematics. Thus central backgrounds concerning these topics will be described briefly in order for a better interpretation of the empirical results of the study.

Theory of Mathematical Thinking Styles (MTS)

The term mathematical thinking style (MTS) is characterized as follows:

"A mathematical thinking style is the way in which an individual prefers to present, to understand and to think through, mathematical facts and connections by certain internal imaginations and/or externalized representations. Hence, a mathematical style is based on two components: 1) internal imaginations and externalized representations, 2) on the wholist respectively the dissecting way of proceeding." (Borromeo Ferri 2004, 2010)

A central characteristic of the construct mathematical thinking style is the distinction between abilities and preferences. Mathematical thinking styles are about how a person likes to understand and learn mathematics and not about how good this person understands mathematics. This approach is based on the theory of thinking styles of Sternberg (1997). So in the sense of Sternberg (1997), "A style is a way of thinking. It is not an ability, but rather, a preferred way of using the abilities one has." Mathematical Thinking Styles were reconstructed qualitatively and currently quantitatively measured with school and university students and teachers (Borromeo Ferri 2014, 2015). Thus the theory of Mathematical Thinking Styles is well-grounded theoretically and empirically. The three main styles are described as follows: Visual Thinking Style: Individuals prefer internal iconic representations and externalized iconic representations as well as the holistic way of proceeding. Analytic Thinking

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Style: Individuals prefer internal symbolic representations and externalized symbolic representations. They like to understand mathematical facts or solve problem step after step. Integrated Thinking Style: Individuals combine visual and analytic ways of thinking and are able to switch ways of proceeding.

Teacher education in mathematics

Results of the COACTIV-Study (Baumert, Blum, Neubrand) and the TEDS-Study (Blömeke, Kaiser, Lehmann) showed how amongst others that content (CK) and pedagogical-content knowledge (PCK) of becoming mathematics teachers are different dependent on the teaching degree. The becoming high-school teachers in Germany for example achieved a high score in both CK and PCK compared to all the university students who studied the other teaching degrees (Baumert & Blum 2010). However, internationally the becoming primary teachers in Taiwan and Singapore got the best results in CK and PCK, whereas Germany's primary and also secondary teachers were in the middle field (Blömeke, Kaiser & Lehmann 2010). So the differences between the teaching degrees became apparent and have to be considered. Beneath these findings studies from the perspective of higher-education mathematics of the last years show the problems of first semester students with basics in mathematics. Often there is a discrepancy between suitability and conception concerning studying mathematics (Roth, Bauer, Koch, & Prediger 2015). Most of the becoming teachers belief that the level of mathematical content they learn at university is too high with regard to teach students at school. Nevertheless the students who are interested in mathematics and want to be a mathematics teacher still have to work hard to get their final exam. Although mathematics is taught mostly conventionally on the black board and exercises have to be done in a group there is no empirical evidence how students like to learn and to understand mathematics. Until now there is much knowledge about mathematical beliefs in this field, because it was and it investigated often within large-scale studies, but we know less about mathematical thinking styles of university students.

Research questions

- Are there differences between visual, analytic or integrated thinking styles depending on the teaching degrees of the university students? In particular: Do high-school teachers prefer stronger the analytic thinking style than the primary teachers?
- Which personal ideas about how mathematics should be taught do university students' have?

Methodology of the Study

The sample of the study was N = 219 becoming math teachers of University of Kassel and University of Hamburg in their third year of university (87 primary teachers, 67 secondary school teachers , 56 high-school teachers, 11 vocational school teachers). The Mathematical Thinking Style questionnaire with 27 items was used, which comprised four different subscales rated with likert-scale from 1-4, which means from [1] strongly agree to [4] strongly disagree. Also three problem solving tasks (open format) were integrated in the test and therefore a coding manual was developed concerning kinds of presentations and ways of proceeding for solving these tasks. The questionnaire also contains scales from PISA (PISA-

Consortium 2006), in particular scales of beliefs, self-efficacy, motivation, emotion and concerning exercising mathematics. One open question asked students concerning their wishes how mathematics should be taught within lectures and seminars. The data were analyzed with the software SPSS. A special analysis of the means was done in order to determine the characteristic value: $1 \leq x \leq 2$ visual thinking style, $2 < x < 3$ integrated thinking style, $3 \leq x \leq 4$ formal/analytic thinking style. The open question was analyzed deeply and coded in categories concerning the answers.

Selected results of the study

Due to the limit of space only selected results are presented. The comparison between analytic and visual thinking styles showed a decrease of the visual thinking style preferred strongly by the becoming primary teachers to the high-school teachers preferring more the analytic thinking style. While the visual preference is decreasing the preference for the integrated thinking style is increasing along the teaching degrees and the combination between working dissecting and wholistic is preferred by 80% of the participants. Although a mathematical thinking style is a personal attribute it shows that the accomplished teaching degree influences if analytic or visual thinking is preferred most. Becoming high-school teachers have more lectures and seminars of different mathematical topics than primary teachers Germany. So learning and practicing mathematical ways of thinking and notations is more exercised. Similar to the results of previous studies of MTS the integrated thinking style is preferred strongly. The correlation between best marks of school students and preference for the integrated thinking style was significant. Based on the MTS theory this style offers flexibility in thinking when working on mathematical problem independent, if these tasks are presented more visual or analytic (see Borromeo Ferri 2015).

The results of the open question to the university students concerning their preferred way how mathematics should be taught in lectures and seminars was very interesting and leads to discussion. 129 of the 219 participants answered this questions and every respond was analyzed and coded. Finally 8 categories could be reconstructed and the ranking shows that wishes of students to get more concrete examples (85%) during their studies is on the top.

Examples	Visualization	Linking-to-practice	Variety (teaching methods)	Exercises (to be revised)	Script (missing)	Dissecting the content Of lectures	Formalizing
45, 5%	23,5%	8,6%	8,6%	5,9%	4,3%	2,1%	1,6%

Table 1: Preferences and wishes of university students for mathematical lectures and seminars

The detailed comparative analysis between the different teaching degrees concerning each category gave insight into the needs of the students. The demand of getting more "examples" was of high importance for the becoming secondary teachers (45,5%). The category "Visualizations" comprised the wish of students that mathematical contents should be taught more through visual representations like sketches or drawings and was ranked on place 2. Interesting was the discrepancy of this wish between the secondary teachers (52,9%) and the vocational teachers (12,5%). In particular the primary teachers claimed the "Linking-to-practice" (24,4%), because this category describes the individual's noticed profit for the work in school later on. No high-school teacher expressed wishes in this direction.

Also the further categories showed a discrepancy between the teaching degrees. Especially both frequent mentioned categories “Examples” and “Visualization” should be discussed regarding to the higher-education mathematics perspective.

Discussion – to defy conventions?

Looking at the becoming math teachers from the MTS-perspective it offers an insight about their preferences how they like to learn and understand mathematics. Taking the manifold efforts like creating new and specialized lectures, training of tutors or using social apps within lectures and also research in this field into account, there is one aspect, which is not easy to solve: reducing of typical mathematical notations and formalizations. The nature of mathematics as a scientific discipline implies correct and formal ways of notations, which are conventions for a common understanding and language within teaching and learning of mathematics worldwide. At the same time these techniques and the notations are abstract and need time to understand. Using visualizations or concrete examples instead can be supporting elements, but not a substitution. Only 1,6% of the participants and in particular becoming high-school teachers demand for using formalization as a central part of mathematics lectures. Finding a balance between not neglecting mathematical conventions and preparing mathematical contents visually or concretely for offering students a better and deeper insight in mathematics can be a reachable goal. To defy mathematical conventions would be provocative. Besides all the upcoming teaching modules for mathematics for teacher education as well for the undergraduate or graduate level defying the conventions should not be a goal, although it is students’ wish. Recognizing their demand and going a well-balanced way including students’ interaction and ideas can be fruitful for motivation and learning of mathematics at university level for both students and lecturers.

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A CAT's glance towards abstraction

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Basic mathematics courses often belong to the top challenges for first semester students of economics and other non-mathematical disciplines. Non-adequate studying and working techniques are essential reasons for that. As a remedy, we introduced an in-teaching system of methodological support called "CAT". Empirical evidence shows that CAT meets the students' needs and also can provide improved performance. On the other hand, not all of the students' difficulties in coping with mathematics could be addressed so far. Some of these difficulties are related to abstraction processes. We present some initial considerations about these processes, aiming at further possible improvements of the support for students.

Introduction

The author's long run experience in teaching basic mathematics courses for economists indicates that one of the main reasons for the students' problems in coping with mathematics originates in non-adequate techniques both of studying and of mathematically working. As a remedy, from 2010 on we introduced an in-teaching system of methodological support called "CAT". Since then, CAT could be gradually improved grace to the results of accompanying empirical studies, which have been supported by the khdm, and grace to the immediate feedback from the students and the members of the teaching team. A detailed account to key features of CAT can be found, e.g., in Dietz (2013, 2015). Moreover, main findings of the empirical studies are presented in Feudel and Dietz (2015). In particular, the studies show that CAT meets many students' needs and that improved academic achievements in the group of students with medium initial skills are rather likely due to CAT. In addition, the studies indicate that CAT's self-assessment support should be given more attention. However, some of he student's difficulties are related to *abstraction processes*, which have not yet been sufficiently addressed. We present some initial considerations of this problem, aiming at a better support for the students.

A little more about CAT

CAT combines a teaching and studying *philosophy,* working *procedures*, and „*product guides*". Key elements of its philosophy are the principle *'aid for self-aid',* and both the requirement and support of a *conscious knowledge management*. The acronym CAT itself is derived from the procedures: *check-lists* (providing instructions and reminders for regular working steps), *Ampel* (german for traffic lights, supporting self-assessment), and *toolbox* (giving support for problem solving).
The most significant role is played by the check-list „reading". It provides procedure-like instructions how to read mathematical texts appropriately – from 'spelling' symbols, notions, and formulas, up to he construction of valid mental concepts. The students are encouraged

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to keep track of all important symbol and notion definitions in an own *vocabulary* and to augment its entries step by step by consistent extensions like examples/non-examples, visualisations (if approporiate), related statments, applications, etc. The collection of all these extensions together with the vocabulary entries is called *concept base*; it is essential for building a valid mental concept image in the sense of Tall & Vinner (1981). A detailed account of the steps of the reading process is given in the course textbook (Dietz, 2012).

The need for abstraction

Certainly, there is no need of discussing the general role of abstraction within mathematics. Moreover, it is known that many mathematicians gained their ability to deal with abstract objects, so to say, automatically during their education, without abstraction itself being an explicit educational subject. However, there is much disagreement w.r.t. how far *non-mathematicians* should be able to perform abstraction and to understand and use abstract concepts. Accordingly, there appears to be no common consensus w.r.t. to the question whether, and how, abstraction can be taught. Note that, for the moment, we use 'abstraction' in a rather broad sense; in particular, we cover mental activities of understanding abstract objects, of working with them, or even creating such. In the author's opinion, there is a particular need for supporting such abilities. Here are some indicators for that:

- Following Piaget (e.g. 2003), abstraction abilities form an intrinsic feature of higher cognitive development. Hence they should necessarily be supported by *any* education.
- Good study results are closely related to the students' metacognitive abilities to organize their own processes of studying, working, and problem solving. On the one hand, these processes involve a vertical reorganization of structures, which can be seen as a particular aspect of abstraction (in analogy to Hershkowitz et al. 2001); on the other hand, the support of such processes is CAT's concern.
- In modern economics, there is a particular demand for abstract approaches in order to understand the principles of economic phenomena in a qualitative way, irrespectively of specific quantitative assumptions. E.g., rather than determining the operating minimum of a specific cost function like $x \to x\text{^}3 + 2x\text{^}2 + 111$, x nonnegative, by laborious calculations, students should be able to specify the operating minimum of *any* cost function, given only that it is of neoclassic type. Note that here we have to deal with abstract objects both from economics, and mathematics. Thus we have to establish the correct correspondence between these as well.

A little more about abstraction

There is a rich literature about ‚abstraction', providing an a broad variety of concepts which differ in various aspects. From the point of view of describing underlying cognitive processes, Piaget contributed a well-recognized cornerstone (e.g., Piaget 2003). Meanwhile, Piaget's concepts of empirical, pseudo-empirical, and reflective abstraction have been augmented by further abstraction concepts like structural, operational, and formal abstraction, see Tall (2013). With regard to didactical implications we want to mention the initial works of Dawydow (1977) and of Hershkowitz, Schwarz, and Dreyfus (2001). Dawydow (1977) writes „ ... der Abstraktionsprozess besteht darin, die Unabhängigkeit des Zustands ... eines Gegenstandes von bestimmten Faktoren zu bestimmen ... dieser wird gedanklich durch einen anderen ersetzt...“[1]. Hershkowitz et al. treat a contextual theory, with abstraction as an " ... activity of vertically reorganising previous constucted mathematics into a new mathematical structure ... " As can be seen from these statements, the relation between the respective concept of abstraction itself and possible didactic interventions for enhancing the corresponding abstraction abilities is by no means obvious. Generally, as was pointed out by Lowell (1979), the term abstraction has „ ... a wide variety of rather vague uses...“, and even „ ... his [Piaget's] theoretical considerations fail to provide any operational definition fort he term.“ Lowell's concern was that „ ... without an operationally acceptable definition for abstraction, it is difficult to see how any refined analysis of ... human learning in general, can be achieved.“ But notable progress was made since Lowell's complaint. Steiner (1994) exploited the theory of semantic networks and focused on fostering algebraic-mathematical networks of (poor) 10th grade level students. He found out that different kinds of systematic treatment lead to measurable improvements in manipulating fractions, factorizations, and combinations thereof. Hefendehl-Hebeker and Rezat (2015) exposed essential features of advanced algebraic thinking; let us emphasize (1) *a systematic use of variables*, (2) the transition from operational to *relational thinking*, and (3) the development of a *sense for term structures*. The latter includes, briefly spoken, the ability of – or feeling for –

- (3a) sensefully clustering subterms of an (algebraic) expression
- (3b) appropriately reading symbolic expressions *to extract all information*
- (3c) appropriately choosing symbols.

Most of these abilities describe particular aspects of abstraction, partly augmented by additional thinking operations.

Outlook

We are interested in treatments on a metacognitive level as parts – or possible extensions – of CAT, repectively, in order to enhance particular abstraction abilities oft he students. Doing so, we have to confine ourselves to those aspects of abstraction that are accessible to rule-based instructions. Ideally, these could be parts of a suitable checklist.

At first, we note that the forementioned item (3b) is already at the very heart of CAT's check-list "reading". Up to now, the focus was on a deep understanding of mathematical concepts, starting from the respective concept definition. Obviously, widening the focus by systematic training of CAT's reading technique in the domain of problem solving promises to provide an enhancement of these particular abstraction "skills". Further aspects of abstraction that promise to be accessible for systematic training are

- consequently *symbolizing* given (numeric) values, related to (1)
- *structuring and (re-)symbolizing* mathematical expressions with the aim to recognize object categories of higher order, related to (3a); to give an example:

[1] Author's English translation: ...the process of abstraction consists in determininig the independence of a subject from certain factors ... this [subject] is mentally replaced by another

$$e^{2\sqrt{x}} \ldots = e^{2\sqrt{x}} \ldots = e^{something(x)} \ldots = e^{\varphi(x)}$$

(Instead of working with a particular functional structure of the power a transition to a more abstract functional expression is achieved, making general derivation rules more easily feasible.)

- application of *recursive techniques*, to give an example:

Given arbitrary sets A and B, simplify the following expression:

$$A \cap (B \cup (A \cap (B \cup (A \cap B)))).$$

In analogy to Steiner's work, it appears to be promising better abstraction skills of the students by integrating these techniques in the set of methodological instructions and by training them properly. The developement of details, however, is subject to future work.

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Reducing math anxiety

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The common discourse in mathematics insinuates that mathematics is concerned with special objects and their properties which are studied via their representations. Thereby the used language is similar to that in physical sciences and gives the impression that mathematics gives us exact descriptions of the respective objects. Yet there is an unsurmountable tension in that those objects are so called abstract ones not amenable to our senses. The thesis now is that this leads to essential problems for the learner. As a way out a non-descriptive interpretation of mathematics according to Wittgenstein is proposed which also should be offered to the students. Therein math is viewed as a system of norms and rules which can be used as models and descriptions.

Diagnosis

This contribution is concerned with potential problems of students of mathematics at universities regarding ontological and epistemological aspects of their learning. It is a widely shared experience of university teachers that many students fail, show signs of anxiety and uneasiness and have great difficulties of organizing their learning and working in math, especially so regarding the understanding and constructing of proofs. Often the teachers are blamed for bad teaching or the mathematics is blamed to be meaningless for the students except for a small minority of so called highly gifted ones. In school mathematics the situation in principle is not much different but it is alleviated by a strong orientation towards applications and practical problems which partly succeed to give meaning and relevance as well to math even if it is experienced as very difficult. The learners there get the feeling that math is talking about and describing situations in reality like in physics or in economics. Thereby math becomes more like a technique, a collection of methods for solving problems and the question of genuine mathematical objects does not arise. Didactically this approach is exploited by trying in an empiricist way to develop (also cognitively) the mathematical objects by abstraction out of concrete situations.

Yet, in modern math classes like calculus or algebra at the university this reference to topics outside of math is lacking even if it is used for motivating the pure math. The standard discourse in classes and textbooks presents math as the science the objects of which are sets, numbers, functions, algebraic structures, various kinds of spaces and so on. About those mathematical objects theorems are formulated and proved which are taken to give us absolutely true and unchangeable propositions about properties of the objects. Mathematical theorems are stated like statements in natural sciences and on the face of it there is no doubt that the objects concerned can be viewed in analogy to physical objects, a position taken for instance by Gödel. But it is also generally agreed that the objects of mathematics are not located in space and time and that they therefore are not accessible to the senses as are at least most of the physical objects. Thus a paradoxical situation arises: in math so

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called representations (formulas, diagrams, graphs, etc.) are used in all kinds of mathematical activities and they are considered as a kind of description of the inaccessible objects themselves. Because of the latter quality one never can scrutinize the fidelity of the description and one is without any alternative restricted to the representations themselves. This dilemma of a factual discourse about objects which in principle cannot be used either to justify or to falsify the assertions made about them in theorems is in principle unsolvable. It is my thesis that the talk about objects and their properties in pure math is a potential source for a feeling of lack of understanding and for a deep uneasiness on part of the students. From most other areas of learning they bring the experience that the respective discourse has reference and meaning somewhere in the perceivable reality in stark contrast to what they experience in math which thereby is in danger to be seen as meaningless.

A first example is presented already by the "ideal" objects of geometry: geometry does not talk about the figures on paper but about "exact" ones. For the latter the question of existence and localization has belabored philosophers and mathematicians since centuries without any conclusive solution. Any kind of (actual) infinity poses similar unsolvable problems. How to understand the talk about "all" natural numbers which yet is indispensable for understanding notions like (infinite) sequence, limit, irrational number and so on? Cantorian infinite sets are likewise prone to arouse similar questions. Here one easily has the impression not to know about what the mathematical discourse is speaking. Nobody can survey an infinite set and the intended referents of set theory remain hidden completely. But still, in the common textbooks sets are presented as objects of the mathematical investigations much like, say, cells are the objects of biology. Sometimes platonistic interpretations are offered (e.g. in Deiser, 2010) but mostly the common way of presenting math is seemingly viewed to be completely unproblematic. The students simply have to get used to that way of mathematical behavior but – my thesis – for many of them this is in fact not much more than induction into a meaningless operating with signs and words based on memorizing. Thus, what is needed is an alternative to the widespread physicalist or every day interpretation of mathematical language according to which words and signs receive their meaning by referring to something outside the language itself. For all the philosophical aspects here and in the following a good reference is Shapiro (2000).

Therapy

The main obstacle for a sober and not metaphysical interpretation of mathematical discourse appears to be the question about the ontological status of mathematical objects from which derive related epistemological issues. Various are the trials by philosophers to solve this problem none of which is satisfying because in one way or the other all stick to mathematical objects be they platonistic, empirical, mental or fictitious ones. The only exception and substantial alternative appears to be the view proposed by Wittgenstein which we will consider here now. This view is embedded into a general philosophy of language and of meaning. Wittgenstein proposes to analyze meaning of words and more generally of signs not in terms of reference but of the usage made of them in the practice of sign use. The central notion for this analysis is that of language (or sign) games which are considered as the locus of meaning which therefore is shifted from the outside to the interior of sign use. If at all, and what, signs denote results from the "moves" (or language acts) within the sign game. The latter is a practice governed by explicit and implicit rules which are to be learned by participation in the respective game. The signs play a role analogous to that of figures in certain games like chess. To be meaningful there need not exist external referents for the signs of the game but there often are rules for how to use the signs to designate objects outside of the game. This is not the case for chess but essentially so for the sign game of arithmetic as Wittgenstein emphasizes. A figure in chess receives its meaning not by designating something (a king is not referring to a "king") but by the rules governing how to move it. Nevertheless we use nouns for speaking about the figures of chess and Wittgenstein views the use of nouns in math in analogy to such practices. This is to say, the meaning of a mathematical term is not fixed by its reference to a mathematical object of what kind ever (like in Frege) but by the rules for operating with it (like in Heine and Thomae). This, for instance, very nicely solves all ontological qualms regarding complex numbers where the search for referents plagued mathematicians and philosophers for centuries and of course the students of today. This similarly applies to (pure) arithmetic, or to quaternions, or to finite geometries, etc. The geometric model for the complex numbers here appears just as another sign game which mathematically is isomorphic to the original complex numbers as algebraic entities. It is important to realize the central role played in all these examples (and possibly everywhere in math) by the rules for operating with the symbols, diagrams and terms. That math can and should be interpreted as not being descriptive of a realm of independent objects is even more distinctly underlined by the notion of infinite set as given by Dedekind and Cantor where the ontological issues are even more exacerbated. The common definition (bijective mapping onto a strict subset) should be viewed not as describing an essential property of infinite sets but as prescribing a way of using that term, of offering a rule for how to speak "about" infinite sets even if there are none of them independent of that speaking. Of course this rule for the game of set theory is embedded into a vast collection of other rules about how to use terms like set, mapping, bijective, etc. From all these rules and conventions the game is developed further which I propose to view in analogy to the development of a fiction or of a theater play where again certain rules are to be regarded. Which those rules are is open to negotiation as the example of the axiom of choice shows very clearly where it took some time until the mathematical community agreed to use this axiom as a powerful means for proving theorems. One has for that investigated where and for which purpose the axiom has to be used and what will be "lost" if it is not admitted. This somehow formalist approach does not deny the role played by intuitions or by various heuristics. Like a good chess player has available a great many experiential guidelines though the game itself is completely controlled by formal rules.

We have mentioned several times the notion of rule to which Wittgenstein accords great importance. He proposes to analyze mathematical concepts and theorems as a kind of rule which would save us the problem of truth in math. An illustrative example is the notion of sphere which then is not taken to describe some ideal objects. The mathematical sphere is not an object, even not an ideal or abstract one, but a rule for judging objects regarding their sphericity. If an object complies with this rule (more or less) we will consider it as a sphere. From this rule again other rules can be deduced which further illuminate our use of the term "sphere". In this vein, a theorem like that about the "infinity" of prime numbers is not asserting the factual existence of infinitely many objects but it is just another rule within the game of arithmetic. This interpretation of (pure) math as a collection of rules and norms sidesteps problems like those about the necessity, the certainty, the eternal validity of mathematical theorems since those issues for rules simply make no sense. For more on Wittgenstein see Dörfler (2014) and the references there.

I think that students might profit if one would discuss such matters explicitly with them offering thereby ways for a sober and not mystical way of thinking about math. Of course this is not meant in a dogmatic way or as the ultimate solution of all respective problems but in a good Wittgensteinean sense as a possibility to think about mathematics. The approach outlined above shows that the traditional questions about mathematical objects can be circumvented or even be shown to be void problems. Rules and norms do not have a truth value but they have to be accepted and followed by a community; they also cannot be falsified but only become outdated or appear to be no longer adequate. Thus the miracle of the apparent eternal necessity of mathematical truths dissolves. Here again the comparison with chess is very helpful. Understanding math no longer resides in a mystical access to otherwise inaccessible objects but in the conscious acceptance of the rules. The system of rules in math is clearly very complex and partly implicit such that this view does not spare the learner great effort and precise attention to the rules. But she now will take part in a public activity with signs and not in secluded cognitive endeavor. Learning math turns from a mental construction of mathematical objects to the social participation in a practice which is guided by rules and conventions which have to be accepted by the learner to be experienced as meaningful. The rules and norms are in principle conventional but they often derive from practical mathematical and not mathematical demands and situations. And, as Wittgenstein emphasizes, they have to prove useful outside of mathematics. By this view a very sharp distinction is drawn between math and all sciences which might be very important for a relaxed understanding of math. Whereas in the sciences always an object can be assumed about which the science is reasoning but which is not itself present neither in the classroom nor in the textbook, in math the whole story is on the blackboard or on the pages of the books. There is nothing behind or under the mathematical "text", a proof is not about a proof. It is again like with chess where everything is clearly and publicly visible on the board. Thus a last consequence would be a strong focus on the mathematical signs (formulas, diagrams but also verbal expressions like in set theory) and the operations with them.

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How can Peer Instruction help the students' learning progress?

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In a case study I will explore the learning process and how students acquire new knowledge during the discussion. The paper focuses on peer discussions initiated by a multiple choice clicker question that addressing the mathematical language.

Introduction

Mazur (1997) recommends the usage of clicker questions[1] during lectures as follows: First the lecturer presents the question and the students vote for the first time, then they discuss their vote for a few minutes with their neighbours and vote again on the same question before the correct answer is presented. Mazur named the peer discussion Peer Instruction (PI).

However, some lecturers who use clicker questions skip the PI and explain of the correct reasoning after the first vote. They think a clear and accurate explanation will lead to more student learning than an explanation by peers would (Smith et al., 2009, p. 124). Smith et al. contradict this opinion. They point out that research in physics has shown that instructor explanations often fail to produce gains in conceptual understanding. Moreover they have shown in an undergraduate genetics course for biology majors "that peer discussion can effectively promote such understanding" (Smith et al., 2009, p. 124).

During PI, students try to find the correct answer. Many lecturers assume that students can benefit from peer discussion only if someone in the group initially knows the correct answer and reasoning and can instruct the rest of the group (Smith et al., 2009). When I talked to mathematicians this is a frequent objection to the use of peer discussions in first year courses. Many of them doubt that enough students have the knowledge to convince the others. So in this paper I want to show how first year students work together during PI and how they benefit from the discussion.

Learning through Discussion

According to Miller interaction is the key to developing new knowledge. During a discourse like in PI a social conflict ("who has the right answer") can lead to a cognitive dissonance ("which is the right answer if both have different meaning and both think they would be right") (Miller, 2006). This dissonance can be a starting point for gaining individual knowledge.

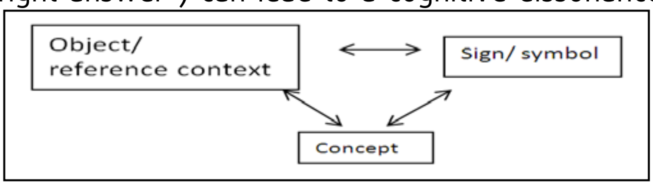

Fig 1: Epistemological triangle

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[1] You can find one example of such multiple choice questions in this paper.

Steinbring supplies a “theoretical basis, where the epistemological conditions of mathematical knowledge are particularly related to interactive constructions of knowledge”. (Steinbring, 2005, p. xii). He combines the epistemological triangle as seen in figure 1 with Luhmann’s concept of communication.

In interaction with others the students produce actively reciprocal connections between the „points" of the triangle (Steinbring, 2005). For example when students discuss about the concept of functions they relate the sign/symbol “f” with a diagram as a reference context. But this relation is not fixed; it can be modified during the interaction with others. So the “epistemological triangle reflects the particular status of mathematical knowledge as it has been constructed in the interaction to a certain point of time” (Steinbring, 2005, p. 78).

This view of producing mathematical knowledge through interaction allows us to model “the nature of the (invisible) mathematical knowledge by means of representing the relations and structures constructed by the learner in the interaction” (Steinbring, 2005, p. 23).

Methodology

As mentioned above I would like to get a deeper insight into students interaction during Peer Instruction. Case studies can provide a rich and significant insight into events and behaviours, provides descriptive details about a particular phenomenon, can increase understanding of phenomenon and explore uncharted issues (Yin, 2006).

In this paper I will present a case study on the students’ discussion process for one clicker question and concentrate on a few group discussions. According to Yin’s (2006) classification this case study is a one case study with “embedded” subcases.

The results presented here are part of a larger study in an undergraduate analysis course with 16 questions in 4 theatre style lecturers each 90 minutes long. The clicker question (figure 2) that is focused on in this paper was presented at the beginning of the second lesson.

With the given clicker questions the students had approximately one minute to think about it on their own before they voted for the first time. Afterwards they had approximately six minutes to discuss the solution with peers and vote a second time, followed by the lecturer’s clarification. For analysing the Peer Instruction, the students were asked which group is willing to record their discussion by dictaphone. Six groups volunteered.

For good validity and reliability of the case study the audio recordings of the discussions were transcribed using GAT rules (Breidenstein, 2004). Afterwards the transcripts were interpreted turn-by-turn analyses among members of the study group as described by Krummheuer (1992). In order to uncover the knowledge construction, they were then analyzed with Steinbring’s epistemology oriented methodology as describe above.

The clicker question

Definition of a local Maximum

Let D be a subset of $\mathbb{R}$ and $f: D \to \mathbb{R}$ with $x_o \in D$:

Please choose the definition that fits best to your understanding of a local maximum:

f has a local maximum at x_o if and only if,

A) for all $\varepsilon \in \mathbb{R}^+$ and all $x \in D$ with $|x - x_o| < \varepsilon$ there is: $f(x_0) \geq f(x)$.

B) there exists an $\varepsilon \in \mathbb{R}^+$ so that for all $x \in D$ with $|x - x_o| < \varepsilon$ there is: $f(x_0) \geq f(x)$.

C) for all $\varepsilon \in \mathbb{R}^+$ there exist a $x \in D$ with $|x - x_o| < \varepsilon$ and $f(x_0) \geq f(x)$.

Fig 2: Voting question

Based on the findings of Dubinsky and Yiparaki (2000), that many students have major problems understanding the interlacing of "for all…there exists" (AE) and "there exist…for all" (EA), the clicker question presented in figure 2 was designed.

In this question, the correct answer B) is contrasted by the two definitions A) and C). In definition A) the students should realize that "all$\varepsilon \in \mathbb{R}^+$" and "all $x \in D$" can be shortened into „to all $x \in D$" thereby defining an absolute (global) maximum. Definition C) instead does not fit the idea of a maximum because each point of the function fulfils the prerequisite.

Results

At the beginning of the discussion none of the students could correctly reasoning that definition B) was right. The discussions revealed many misinterpretations like the following example.

At the beginning definition A) was the favourite definition for many students. These students interpreted the statement (sign/symbol) "for all $\varepsilon \in \mathbb{R}^+$ and all $x \in D$ with $|x - x_o| < \varepsilon$" as an ε-neighborhood in the reference context illustrated in figure 3. One student reasoned his interpretation by referring to the definitions of the convergence of sequences in which ε is used as arbitrary small number.

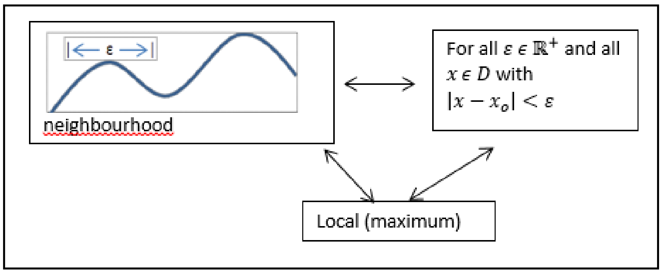

Fig 3: often seen interpretation

S: *Definition A) mostly makes sense for me because it means that you approach over all x (2.0) let the interval getting smaller and smaller.*
S: *I see a connection to the concept of convergence (.) that you shorten the distance more and more (1.0) and nevertheless the $f(x_0)$ is the greatest.*

In other groups students used the ε-δ- definition of continuity to justify the correctness of definition A). It seems that this common use of ε leads to the meaning of the sign "for all $\varepsilon \in \mathbb{R}^+$ as seen in figure 4.

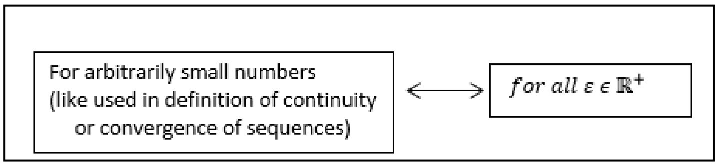

Fig 4: often seen interpretation

Altogether this misinterpretation could be found in three out of six groups.

Two of these groups were able to recognize their mistake and understood that this definition A) is a definition of a global maximum. So a conceptual change from fig. 3 to fig. 5 could be seen. The discussion process of these two groups were even more successful. They could even give a convincing argumentation for definition B) to be correct. For example, one student who was asking his classmates "are there any differences between definition B) and C)" at the beginning of the discussion was then able to convince his colleagues at the end that definition B) was right with the words:

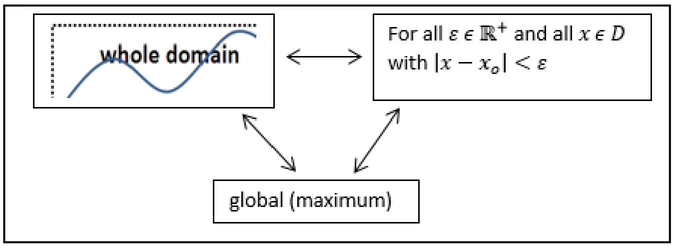

Fig 5: correct interpretations

> *L: you find any interval around x_0 so that every function value is smaller (-) then you have a local maximum and that is exactly what is stated in b (-) find an epsilon interval around x_0 and all x must be inside this (-) that is exactly what is formulated in b*

A learning progress in understanding of the meaning of the mathematical expressions could be observed in five out of six groups. Only the sixth group did not even try to interpret the three definitions. They were afraid to say something wrong.

The key for the learning progress was mutual support. They helped each other by asking and answering question, giving critical comments, revealing gaps in the chain of argumentations, supporting argumentations and the famous gradual generation of thought through talk. This collaboration was possible because all statements were taken seriously. The attempt to find connections between the mathematical symbolic expressions and their visual imaginations especially helped the students to overcome misunderstandings and supported the construction of new knowledge.

Conclusions and discussions

The examples show that peer discussion provides a chance for a learning progress and that students are able to reveal misunderstandings and successfully change their misinterpretations of mathematical expressions. It also shows that students work together in a collaborative way, so the name Peer Instructions is misleading. It gives the idea that high achieving students support lower ones by explaining the right answer. But such explaining was not been seen here. Instead collaborative teamwork was the key for creating new knowledge. So I recommend using the words peer discussion or peering learning instead of Peer Instruction. To determine the best conditions for good collaboration further research is needed.

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Connections: mathematical, interdisciplinary, personal, and electronic

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How can creating links improve our students' learning experience? We argue that students' appreciation of the intellectual links profoundly affect their grasp of the material. We argue that personal links have a major effect on their confidence and motivation. We analyze a range of different types of links, characterizing each and identifying its benefits, and talk about how each might be created. Lastly we consider how the landscape will change as courses move online.

Why Connections?

For most people, learning involves making connections: Connections between ideas, connections between fields, and connections between people:[1]

> *"Stunning new research on the brain by neuroscientists is adding a new dimension to our knowledge about learning that reinforces our previously tentative conclusions from cognitive psychology. This research provides growing evidence that* ***learning is about making connections"*** *(Cross, 1999)*

How does the goal of building connections inform our teaching? In this talk we consider how to build different types of connections and the benefits of each.

Mathematical Connections

The most salient connection is probably that between different areas of mathematics. For us, these connections are the life-blood of mathematics—they create the depth of understanding that allows us to analyze problems from a flexible point of view. Insight flows from the power to match the point of view to the task at hand. In addition, research shows us that "as students learn a discipline, their knowledge of the structural relationships among parts of the discipline become(s) more like that of experts." (Schoenfeld, 1982).

However, many students do not easily see mathematical connections. Student feedback makes it clear that students' views of mathematics are often not the same as ours. For some students, mathematics is a set of procedures—a point of view that is unfortunately reinforced by the fact that the tests they take can often be done entirely procedurally, even if they were not intended to be that way. While there is general acknowledgement that procedures are important, they cannot be learned effectively in isolation: "To develop procedural fluency, students need experience in integrating concepts and procedures and building on familiar procedures as they create their own informal strategies and procedures." (NCTM, 2014)

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[1] Emphasis in original.

- Challenge: How can we increase students' appreciation of the connections between different areas of mathematics?

The answer lies in the activities students do. The talk will characterize successful approaches to constructing activities and show examples from several courses. (Hughes Hallett, Gleason, & McCallum, 2013)

Interdisciplinary Connections

For students taking mathematics in the service of another field, interdisciplinary connections are essential as motivation and inspiration. In an Indian school where over 80% had failed mathematics, students reported in interviews "they are not interested in studying Mathematics" because they perceive the subject as "too far from life to catch [their] interest". (George & Thomaskutty, 2007)

But interdisciplinary links are not sugar-coating—their role is not to make an unpalatable subject palatable. Their role is to enable students to develop a deeper understanding of both mathematics and the other field. To ensure that this occurs, the applications shown have to be authentic. Contrived examples simply reinforce students' notions that mathematics is not actually useful.

- Challenge: How do we create meaningful interdisciplinary links to fields? Especially with fields that we have never studied ourselves?

The answer parallels many of the recent advances in research—where the cutting edge is frequently interdisciplinary. The National Academy of Sciences reports (Andreasen, 2005) "Advances in science and engineering increasingly require the collaboration of scholars from various fields". In particular, finance, biology, economic development, education, the law all use mathematics increasingly frequently and now depend heavily on data. Our teaching will similarly benefit from interdisciplinary input.

The talk will consider practical ways to get this input and suggest how examples may differ from audience to audience. We will see a range of structures for interdisciplinary links, from co-teaching, to the use of joint assignments, to projects within a math course. These links are easier in some parts of mathematics than others, but all are appreciated by students. The response of faculty in other fields to the meetings described in "Voices of the Partner Disciplines" (Ganter & Barker, 2004) was superb.

Personal Connections

An undervalued tool in our teaching arsenal is out ability to forge personal relationships with our students. Some students do not want anything from a course except the material, but others are questioning the direction of their lives and greatly value our support or critique. Asked to recall a good teacher, most people point to someone who believed in them—not a teacher who did a brilliant job of presenting a theorem. Technical teaching skill is central—but it is only part of the story. Equal enthusiasm for the material and for the students is an enormous asset because "Engaging students in mathematics conversations in classrooms is central to the development of students' skills and understanding." (Webb, 2014)

Yet there are often difficulties: Culture, language, interest may be barriers, and time is always short.

- Challenge: How do we bridge the gaps and structure beneficial relationships with students?

The answer includes listening as much as talking, restricting advice until it is asked for, and knowing who would benefit from our support and who would not. Refer elsewhere the ones that would benefit from other advice. Then the moments spent talking outside class may be some of the most important teaching moments we have.

Electronic Connections

With the rapid growth in online courses, we need to be able to adapt our answers in each of these areas. In the curricula arena, it is not hard to imagine the transition. Mathematical and interdisciplinary materials can be either paper or electronic. Indeed, interdisciplinary videos are likely to be a great improvement over paper materials. The transition to online is likely to occur naturally with time.

Online grading is already well-established and fairly robust. WebWork, WebAssign already carry out a great deal of the day-to-day grading in US mathematics departments and are surprisingly helpful. When the grading of verbal answers and explanations becomes possible, it will be a huge boost for electronic courses.

However, security, which is not important for MOOCs but vital for credit courses and degree programs, is currently a gaping need. The current online proctoring arrangements are largely not adequate for mathematics. The options lag far behind what is needed.

- Challenge: How will the technology adapt to provide security for credit courses?

Meeting thus challenge is unlikely to involve us directly, except as users. But as companies become increasingly interested in providing online training and MOOCs become essential gateways rather than luxuries, the pressure to improve security will likely produce results. As a community we need to be ready with requests that can shape the service into one we can use.

In the realm of personal relationships, an electronic-only connection poses new questions.

- Challenge: How will we create personal connections with online students?

The answer may involve stitching together existing software or the next generation offspring—for example, Facetime, Skype, screen-sharing software—to enable us to see into the minds of our students in the same way as we can during an office visit.

An online medium has huge potential. Instead of spending time on presenting material that students can read or learn from a video, we will be able us to focus our teaching on *teaching*.

- Challenge: Before a paradigm is thrust on us, the community of mathematics educators should shape the electronic connection they want.

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Problem solving opportunities in frontal classes: Inquiry in teaching practices and learning strategies

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The presentation is based on the results of two studies aimed at exploring opportunities for enhancing students' active learning through problem solving in frontal lectures and tutorials of linear algebra and calculus university courses. In the first study,traditional linear algebra lectures and lectures involving Classroom Response Systems were explored. The study resulted in an identification of a set of practices by which experienced lecturers created opportunities for interactive problem solving. The second study explored students' learning strategies in calculus tutorials. We found that while some students followed the exposition, other students periodically stopped listening, engaged in independent problem solving, and then attempted to catch up with the exposition. Implications are drawn.

Rationale

A consistent recommendation made by many scholars is to enhance student learning of mathematics in higher education by collaborative problem-solving experiences (Artigue, Batanero & Kent, 2007; Dorier, 2000; Holton, 2000). This recommendation is frequently interpreted as a "farewell, lecture" call for the deep reformation of traditional university teaching (Mazur, 2010). In spite of this recommendation, and the fact that, generally speaking, findings about the effectiveness of lecturing are not encouraging (Cooper & Robinson, 2000), mathematics is still taught in many universities in a traditional format. By a traditional teaching format, we mean a combination of frontal lectures in which professors explain theoretical material to groups of 200-400 students, and frontal tutorials in which teaching assistants (TAs) explain problem-solving methods to groups of 40-80 students. This format implies that following the exposition and making notes are the main student activities during a class.

The reasons for sustaining the traditional teaching format are related to the logistic constraints of universities (e.g., Walczyk, Ramsey & Zha, 2007), as well as to the preferences and beliefs held by either lecturers or students (Roth-McDuffie, McGinnis & Graeber, 2000). Additionally, there are studies that suggest: (i) frontal classes of outstanding lecturers *can* be engaging to students (e.g., Movshovitz-Hadar & Hazzan, 2004); (ii) the instructor quality, as reflected by the instructor's connection with the students through carefully listening to their questions, probing their understanding, and keeping the level and pace of the course challenging and achievable, is the foremost explanatory variable of the students' successes and failures (Bressoud, Carlson, Mesa, & Rasmussen, 2013); (iii) pedagogies supporting collaborative problem solving are feasible on the small scale (e.g., Yusof & Tall, 1999), but not always on the large scale. Consequently, identification of opportunities to enhance active learning in frontal large-group classes is still an important research enterprise.

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This article concerns two interrelated issues: (1) teaching practices by which experienced lecturers create problem-solving opportunities for their students during frontal large-group classes; and (2) patterns of (active) student learning in such classes. We pursued the first question in context of frontal lectures of a linear algebra course, and the second one in context of calculus course tutorials.

Teaching practices of experienced linear algebra lecturers

The goal of Atrash's (2013) Ph.D. research was to identify and characterize the pedagogical practices that experienced lecturers employ in order to help students overcome their (anticipated) difficulties with the material exposed in frontal large-group lectures (hereafter referred to as PPODs – **P**edagogical **P**ractices aimed at **O**vercoming **D**ifficulties). The research comprised case studies of two experienced lecturers who taught linear algebra at the Technion. For each lecturer two consecutive semesters of teaching were explored. During the second semester, Classroom Response Systems (CRS) were occasionally used as a technological artefact for enhancing the interaction. All the lectures were videotaped. The data consisted of 168 hours of video, field notes made by the researcher during the lectures, and 10 stimulated-recall interviews with each lecturer.

The video material was analyzed in three stages. At the first stage, the episodes in which the lecturers explicitly dealt with student difficulties were isolated in five randomly chosen lectures. At the second stage, the lecturers' PPODs in these episodes were categorized using the grounded theory approach. At the third stage, the identified categories were refined, verified, and implemented in the analysis of 50% of the videotaped material (42 hours of teaching per lecturer). In particular, the PPODs employed in teaching the same topics during two consecutive semesters were identified for each lecturer. The considerations of the lecturers for choosing particular PPODs were categorized in accordance with their reflection on the videotaped episodes considered during the stimulated-recall interviews.

Nine PPODs were identified: *Giving Numerical Examples*; *Highlighting by Asking Questions*; *Making Intentional Mistakes*; *Using Visual Aids*; *Repeating Twice*; *Solving/Proving in Several Ways*; *Anchoring in Real Life*; *Encouraging to Keep Working*; *Advising to Do or Not to Do Something*. The first three PPODs included short breaks (of 10 seconds to 2 minutes) in the exposition during which the students were encouraged to think about the given challenge.

Consider an example related to *Making Intentional Mistakes*. At a lecture on vector subspaces, Lecturer B. wrote on the whiteboard: "Given two subspaces U and W: $U \cap W$ is a subspace, and $U \cup W$ is a subspace." The lecturer then proved the first part of the claim:

"$0 \in U \cap W$ since $0 \in U$ and $0 \in W$. Following this, the set is closed under the operation of multiplication by a scalar: $v \in U \cap W;\ \alpha \in F\ \Rightarrow \alpha v \in U$ and $\alpha v \in W \Rightarrow\ \alpha v \in U \cap W$. Subsequently, the set is closed under the operation of addition as follows: $u, v\ \in\ U \cap W \Rightarrow u, v \in U \Rightarrow u + v \in U$ and $u, v \in W \Rightarrow u + v \in W \Rightarrow u + v \in U \cap W$.

At that point the lecturer provocatively said: "In order to prove the second part of the claim, about the union, I just change the sign." He then inserted ∪ instead of ∩ in the above proof, looked at the students for about 5 sec in silence and declared: "The second claim is not true, where did I make a mistake?"

There were no immediate responses, so the lecturer continued and gave a counterexample: "Take for instance two subspaces x-axis and y-axis. The union of x-axis and y-axis is not a subspace since it is not closed under the operation of addition: $(1,0) \in x$-axis; $(0,1) \in y$-axis; $(1,0) + (0,1) = (1,1)$, and it does not belong to x-axis $\cup$ y-axis." Finally, the lecturer crossed out the "proof" on the board.

In the lectures using CRS the time of dealing with student anticipated mistakes was longer than in the traditional lectures on the same topics. However, the average number of PPODs observed in the lessons with and without CRS was about the same and was not topic-dependant. The most frequently used PPODs were: *Giving Numerical Examples* (97 occurrences in 84 hours of analysed lessons); *Highlighting by Asking Questions* (79 occurrences); *Advising to Do or Not to Do Something* (47); *Repeating Twice* (39).

Given the high load and time constraints of the course, the lecturers made decisions whether to use particular PPODs based on considerations of both a mathematical and pedagogical nature. One consideration was related to the lecturers' mathematical preferences in teaching linear algebra. Namely, one lecturer gave the highest merit to nuanced exposition of theory (e.g., definitions), and the other lecturer saw her main mission in helping students to make sense of problem-solving methods. The lecturers also differed in their beliefs about the students' learning styles and strategies. In particular, one lecturer assumed students learn best when provided with comprehensive explanations, while the other deemed important to enable students to ask questions.

Learning strategies of calculus students in frontal tutorials

One of the goals of the Ph.D. study of Marmur (in progress) is to characterize learning strategies that calculus students employ while listening to explanations by TAs in frontal tutorials. The presentation focuses on the strategies reflected by the students upon an episode in which one of the teaching practices identified by Atrash (2013) had been implemented, namely, Making Intentional Mistakes. The episode concerns the following problem:

Show that the sequence $a_1 = 1,\ a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$ *converges and calculate its limit.*

A design-based research of five iterations (Marmur & Koichu, 2016) has shown that this problem can evoke an aesthetic response from students when a TA acts as follows. The TA begins by attempting to prove on the board the monotonicity of the sequence by mathematical induction. Surprisingly for the students, this attempt fails and the instructor crosses out the proof. Further reasoning suggests that the monotonicity can be proven if one first manages to show that $a_n \geq \sqrt{2},\ \ n \geq 2$. The latter statement, even more surprisingly for the students, again leads to a failed attempt utilizing mathematical induction, and the TA crosses out the content of the board once more. The TA then invites the students to recall the AM-GM inequality (the inequality of arithmetic and geometric means) whose implementation easily accomplishes the proof of boundedness, and consequently the solution to the problem.

Nine students who expressed strong emotions towards the problem took part in individual stimulated-recall interviews. The students were presented a 15-minute video excerpt of the lesson in which the problem had been taught. They were explained that the video served as an aid for them to „relive" the lesson, and were instructed to pause the video whenever they had a particular recollection of what they thought or felt at that moment. The duration of the interviews ranged from 30 to 60 minutes, depending on the level of detail that was shared by the student. The interviews were audio-recorded and transcribed.

One finding was that the interviewed students generally reported the episode of experiencing intentional mistakes in class to be emotionally intense. Referring to entire solution-attempts being crossed-out, student reports included: „I was in real shock", „It was horrible!", „I felt frustrated", and „It was infuriating". Ultimately, the same students also used expressions like „amazing" and „beautiful" to describe the successful solution at the end of the lesson. We utilized the Contrastive Valence Theory (Huron, 2006) in order to explain the occurred changes in the students' emotional responses: a strong contrast between an initial negative reaction-response to a situation, and a later reflective and re-evaluative appraisal-response, is a mechanism that evokes and enhances pleasure.

Another finding was that the strong emotional responses induced by the intentional mistakes seemed to enhance the students' involvement in the lesson. During its course, the students continually produced anticipations and went through a number of emotional „tension-relief" cycles. Moreover, they reported to be active learners in the following meaning: while listening, they attempted to accomplish self-imposed tasks. These included tasks such as: independently testing alternative ideas to the solution; attempting to predict what the instructor is about to do next; identifying difficult places in the proof to come back to later; looking for connections between the problem taught in class and the corresponding homework assignment; and formulating problem-solving strategies from their current experience that they could use in the future. Several students reported they took the opportunity to cope with these self-imposed tasks when the instructor for instance addressed questions of other students. Two students explicitly reported occasionally "disconnecting" themselves from the lesson, looking for a solution on their own, and then coming back to the instructor's exposition. As expressed by one of these students:

> *"I personally started looking for ways how to succeed in continuing from here. It took me about three minutes to disconnect from my thoughts and come back to you. I did look for other ways how to continue and I really couldn't find any. And then I came back. But I kind of had a pause of 'ok, one moment, it can't be that it's impossible to continue'."*

In this study, we utilized an emotionally loaded episode as a magnifying glass under which student engagement in undergraduate mathematics tutorials can be examined. Overall, the findings point out the complexity of student learning strategies whilst listening to a frontal exposition. In particular, there is evidence that students are capable of engaging in problem-solving-like activity during frontal-style lessons.

Concluding Remarks

The traditional teaching format in higher education implies that students' main activities during a lesson are listening and making notes. The main criticism on the traditional teaching format is that it does not encourage active learning (Artigue, Batanero & Kent, 2007; Cooper & Robinson, 2000). We have no intention to downplay this criticism, but rather would like to claim that special effort put into upgrading (instead of radically changing) the traditional teaching format, could align it with the active learning paradigm. As argued, it is unlikely that the traditional teaching format will soon disappear in higher education – it is more likely that new technology-supported means would complement it. Thus, teaching practices encouraging active learning and problem solving in large-group mathematics classes should be identified and disseminated, and the knowledge on what happens with students during large-group mathematics classes employing such practices should be accumulated. Our findings suggest that some of the students manage to engage themselves in problem-solving-like activities not only during the breaks of the frontal exposition, but also concurrently with it. We believe that it is truly important to help regular students develop this skill. After all, learning from a frontal large-size mathematics class can be as active as learning from watching a good movie or reading a good book.

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Perceived competence and incompetence in the first year of mathematics studies: forms and situations

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One issue in university mathematics education is students' motivation. Following recent theories, the personal experience of competence is a very important element for motivational development. We distinguish students' experience of competence from their experience of incompetence and figure out how these experiences relate to different reference norms. For this, we conducted and coded interviews from students' first year at university. The coded items show that students mostly seem to refer to the criterial norm, i.e. standards set through lectures and tasks. Students refer far less to the social or individual norms. However, when they refer to the latter, they much more often report experiences of competence than incompetence. A further analysis towards typical forms and situations shows that in particular the experiences of incompetence relate to factors genuine to mathematics, like the specific language or proof.

Introduction

Studying mathematics is known to sometimes come along with motivational problems, especially in the first year (Daskalogianni & Simpson, 2002). Specifically in German mathematics and higher secondary teacher programmes, motivational problems form one major reason for student drop out (Heublein, Hutzsch, Schreiber, Sommer, & Besuch, 2009), dropout rates ranging from 50% (Heublein, Richter, Schmelzer, & Sommer, 2014) up to 80% (Dieter, 2012). Apart from dropout, students' motivation is also central for their learning. Interest, for example, is an important predictor for the use of deep learning strategies, effort and learning in mathematics (Köller, Baumert, & Schnabel, 2001). We aim at working out reasons for the development of students' motivation in the typical German setting: Rather large groups of one hundred students or more attend lectures on real analysis or linear algebra, which are based on definition and proof. Every week, the students have to hand-in a task sheet, which includes proof-based tasks. The sheet gets marked and is then returned in a weekly tutorial where the solutions are presented and discussed. Typically, only students, who get at least 50% of the maximum score and pass an additional exam, pass the module. Attendance is neither required in the lectures, nor in the tutorials.

Theoretical Background

Following self-determination theory (SDT) by (Deci & Ryan, 1985; 2002), we assume the experience of three basic psychological needs to be crucial to students' motivational development: the needs for perceived *competence*, *autonomy* and *social relatedness*. In SDT, the role of the basic psychological needs for the development of motivation compares to the role of basic physiological needs (food, water) for the development of our body: need satisfaction is necessary to thrive, in particular for the development of interest (Krapp, 2002). A

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major difference is, however, that the satisfaction of psychological needs is a matter of personal perception. Thus, even in the same situation, different persons may experience need satisfaction very differently. We should also note that the perception of need satisfaction is not a matter of conscious knowledge but rather an affective reaction. For the development of motivation with intrinsic qualities, all three needs are important, whereas in our presentation, we will restrict to the need for competence. This need is satisfied when a person feels effective in interacting with his or her social environment and experiences opportunities to express his or her capacities. Since the need satisfaction relates to personal perception, it should not be confused with any external judgement of competence. In fact, people may feel competent at any level of actual ability. In recent years, SDT has been complemented by approaches of distinguishing non-satisfaction of needs from their frustration. Need frustration goes beyond indifference or lack of need satisfaction and results in different need-specific motivation (Sheldon & Gunz, 2009). Students might, for example, not feel competent because no one tells them that their work is good. As a qualitatively different experience, they might even start feeling incompetent if someone comes and tells them that their work is useless. We thus speak of need satisfaction or experience of competence as well as need frustration or experience of incompetence.

In addition, we use categories of reference norm orientation (Rheinberg & Engeser, 2010) for greater differentiation of competence and incompetence experiences. The categories have originally been designed for teachers' judgement of student work and distinguish three reference norms. The *objective* or *criterial norm* is used, when student work is judged based on factual criteria, e.g. predefined learning goals. The *social norm* is used to judge student work based on comparisons between different students (e.g., across the classroom) and the *individual norm* is based on longitudinal comparisons within one student. Since unlike in the situations the reference norms were designed for, the students in our settings have an external judgement of their competence provided by the grading of their tasks sheets and their exams, we add this *external reference norm* as a fourth category. The difference between the criterial norm and the external norm is that the external norm is based on external judgements only, so there is no direct link to factual criteria. The criterial norm, in contrast, builds on mathematical content like topics from the lectures and the tasks.

The research goals for this analysis are: Firstly, what norms do students relate to their competence experience? Secondly, in which forms and situations do students experience competence and incompetence in their studies? The answers to these two questions should provide a basis to discuss two more questions: Thirdly, can we identify aspects of students' experiences of competence, which are specific to mathematics? Fourthly, what could be starting points for improving students' experiences of competence?

Design and Method

Our data is formed by 48 semi-structured interviews with 20 first-semester students. Thirteen students were enrolled in a secondary teacher programme, six of them studied for a mathematics degree and one of them studied physics, and they all attended the same lecture on real analysis. The professor held the course in a rather abstract way and included only few numerical examples. All students were asked to come for an interview in the fourth or fifth week of their first semester. The participants were then asked to come again for a second interview shortly before the end of the first semester and again in the second semester, which they mostly did (20 / 16 / 12 students in the first / second / third interview). The sampling aimed at covering very different experiences, in particular of successful and unsuccessful students. They all agreed in the anonymous, scientific use of their data and were given the possibility to discontinue the interview or delete passages from the tape at any time.

The students were asked to broadly describe their experience and learning behaviour at university. Subsequently, they were more specifically asked for their satisfaction of the basic needs including competence, without explicitly suggesting any reference norm. The interviews lasted 30 to 80 minutes and were taped and transcribed. Since need satisfaction is usually connected to emotions and may thus be remembered quite well, we assume that the students recalled large parts of their important experiences. We thus expect to cover a very broad range of need satisfaction and frustration.

To answer the first question for the reference norms, students' statements in the transcripts were coded by two coders in parallel for both satisfaction and frustration of the need for competence concerning university mathematics anywhere in the transcript. The codes differentiated the four norms (criterial, social, individual, external) mentioned above and were allowed to overlap, which they rarely did. Differently coded items were discussed and unified together with the first author. We want to illustrate the material and our coding by giving some examples:

> *„Usually, it begins when you start engaging with the exercises. And actually it starts in this moment 'okay I have read it, but I don't even understand the task'." (Incompetence, criterial)*

> *„There is something where you realize, ok cool, I got it, apparently the others around don't and the others around account for most likely around 90% of the lecture hall. That's an awesome / well, let's say a small sense of achievement." (Competence, social)*

> *"When she [the professor] wrote the definition and theorems, I did not get it, I had no idea yet. But now that I have calculated a trajectory several times on the task sheets – I don't know why – you then read the definition again and think 'yes it is clear, it makes sense'." (Competence, individual)*

> *"I thought 'well actually, you got / I have done in MY VIEW everything RIGHT on the sheet'. And then you get it marked and there is written '0 out of 5 points'." (Incompetence, external)*

In order to answer the second research question, the coded items were again undertaken a coding procedure, this time more openly since there were no predefined categories. Instead, recurrent themes were constructed from the material. Again, overlapping of themes was allowed and 378 out of the 456 coded items could be assigned to at least one theme.

Results

The distribution of 456 coded items in absolute numbers and percentages of all 456 items is displayed in the following table:

	Criterial norm	Social norm	Individual norm	External norm	∑
Competence	124 (27%)	23 (5%)	44 (10%)	34 (7%)	225
Incompetence	184 (40%)	9 (2%)	10 (2%)	28 (6%)	231

The table shows that the students reported both need satisfaction (4.7 statements per interview on average) and frustration (4.8 statements on average) with balanced frequency. In both categories, they mostly referred to the criterial norm, accounting for two thirds of the statements. The statements within this category were unbalanced towards more reports of incompetence. The social norm was rarely mentioned in the interviews and the elements in this category mainly referred to competence and not incompetence. Similarly, the individual norm, which was mentioned more often, was more often related to competence. Slightly more positive statements were also given based on the external norm.

We should be cautious in generalizing the findings concerning the frequency of codes, since our sampling did not aim at representativeness. In addition, students' reports may be biased and not fully represent their experience and evaluations. The balance of positive and negative items is furthermore affected by students' tendency to put their statements into perspective, so they often added something negative to a positive experience and vice versa, and also the interviewer explicitly asked for both positive and negative experiences.

Anyway, to answer the first research question, we may conclude that students related their competence experience to all four norms and the criterial norm may be predominant. The reference to different norms seems to result in different experiences of competence and incompetence. To give an example, many students experienced problems following the lectures and solving the tasks in the same manner as they did in school and consequently reported competence need frustration referring to the criterial norm. However, when the same students compared themselves with others, who often shared these difficulties, they did not only put their assumed incompetence into perspective but sometimes also evaluated their partial success as competence. For the individual norm, the experience of looking back resulted almost exclusively in perceived competence, e.g. tasks which students did not understand at all seemed to be very simple some weeks later. Students then often realized and positively evaluated their learning and competence.

For the second research question regarding typical forms and situations, the re-coding of these passages for emergent themes revealed four aspects of both positive and negative experiences: "Understanding" (in students' personal sense) the lectures, specific topics and the tasks as well as being able to solve the tasks. In addition, four criteria of success emerged, namely doing these things (1) immediately, (2) on your own, (3) completely, (4) with visible success in terms of e.g. a document or a reaction by others. These criteria are closely linked to students' former learning at school, where good students used to solve tasks on their own and completely. Students' experience at school seems to function as a standard for the evaluation of their experience at university. These standards, however, were slightly adapted in the new learning environment.

Two aspects were specific for positive experiences: a general feeling of all in all mastering the studies and a feeling of having developed a sense for "right and wrong" in university mathematics. Moreover, positive experiences were intensified in four ways: (1) external confirmation, e.g. by a good grading or reaction from peers, (2) by knowing that "what you do is hard", (3) by overcoming obstacles, e.g. doing things you had failed before and (4) explaining mathematics to peers. These positive aspects cannot be seen as independent from the negative aspects, since knowing, that university mathematics is hard and overcoming personal obstacles are two factors which build on negative prior experiences. This is also clear for the positive experiences of having a feeling of mastering the studies and developing a sense for right and wrong, which was taken for granted in school and then questioned in the first weeks at university.

Three aspects were specific for negative experiences, firstly the mathematical language including symbols, the vocabulary and its logic, secondly mathematical proof including techniques and standards of argumentation as well as the purpose of proof and thirdly the work on the task including understanding the aim of the task, possible steps towards a solution and in particular the feeling of being stuck. They represent major characteristics of research-oriented university mathematics (Mason, Burton, & Stacey, 2010; Tall, 1991; 2008).

Discussion

From the methodological point of view, we could see that interviews give a broad range of results on students' experiences. However, the students' spoke rarely about specific tasks or mathematical objects, theorems, etc. It is thus plausible, that there are aspects of studying mathematics, which did not appear in our study, although they are important for students' experience of competence, like the demands of abstract concept formation.

The first research question for the different reference norms revealed that students relate their experience to all four norms. The analysis indicates that for students' motivation, it could be helpful to direct students' attention not only to the criterial norm but also to the social and even more to the individual norm, the latter also being strongly recommended in school (Rheinberg & Engeser, 2010). Possible means could be the provision of individual feedback or tasks which foster students' reflection on their learning.

The second research question for typical forms and situations could be answered in general factors and specific factors for positive and negative experiences. Generally, students' evaluations of their competence seem to be based on their experience from school, where they, unlike in university, were used to understand the mathematics rather quick and could solve their tasks on their own. The positive forms like developing a sense for right and wrong mainly build on prior negative experiences. Negative experiences related to mathematical language, proof and working on the tasks, which strongly differ between school and university. In recent years, they have been addressed in practice-oriented books (Alcock, 2013; Houston, 2009; Mason, Burton, & Stacey, 2010) and in Germany also in new forms of lectures which focus more on proof and problem solving as processes (Grieser, 2013). It is an open and interesting question, in how far these measures may help students to feel competent.

We should keep in mind, however, that some parts of what students experience as serious problems are a natural part of studying mathematics at university and cannot be removed easily. It is not helpful, for instance, to explain the tasks in deep detail: "every task has an explicit […] and an implicit or inner aspect […] If the inner aspects becomes outer (is made explicit) then the whole nature and purpose of the task is lost, just as when you are working on a problem and someone drops a clue or tells you an answer and you feel deflated and uninterested in continuing. Students may then be able to circumvent the work they need to do." (Mason, 2002, p.172). In both the education of teachers and (research) mathematicians, learning and experiencing how to solve problems is a main education goal which requires the students to be stuck in a problem again and again. Therefore, "Probably the single most important lesson to be learned is that being stuck is an honourable state and an essential part of improving thinking." (Mason et al., 2010, p. vii). The students in our study, however, seemed not to have learned this lesson in school. There rather seems to be a change in the didactical contract during secondary-tertiary transition which is strongly connected to students' experiences of competence. Unlike teaching in school, neither the lecture nor the tasks in university are designed for immediate understanding. On the contrary, understanding the tasks is a new task itself. In addition, no one in university would expect an average student to solve every task on their sheet, whereas in school everyone is expected to fully do the homework. These aspects are not just a matter of difficulty, but – as the quotations above show – it is not helpful to explain in detail how the students should solve a task. This is again a major difference between school and university. We believe that more experienced university students do value their experiences based on this new didactical contract, although empirically this is an open question.

We now discuss the third question. As mathematics educators, we are in particular interested in factors which are specific to mathematics. These are exactly the aspects which are specific to negative experiences, namely the language, proof, working on the tasks and the didactical contract as well as the positive aspects building on the negative ones namely overcoming obstacles and being successful despite these problematic issues. Therefore, it would be very interesting to compare mathematics courses across different study programmes to see whether there are differences in competence experiences and in how far they relate to the different mathematical demands.

The fourth and final question asks for starting points for improving students' experience of competence. Since we could see that the individual reference norm as well as the social reference norm are connected to more positive experiences, one could help the students to become aware of their individual learning and its progress as well as to share experiences across the study group. To some extent, one might also try to help students with the specific problematic aspects, e.g. valuing work like exploring a mathematical topic, even if there is no visible outcome on paper or helping them to develop effective strategies for mathematical learning and problem solving. In particular in the context of proof, issues like language and socio-mathematical norms could be addressed more explicitly. Furthermore, the didactical contract could be made more explicit to the students. Although some parts will and need always be hidden, other parts need not. A good tutor would in his or her weekly tutorial not only present solutions but also reflect on these issues. However, this depends on personal commitment. An institutionalised solution might be given by drop-in centres for students (Croft, Harrison, & Robinson, 2009), although it is an open question in how far they may help students feel competent and motivated.

From the longitudinal perspective, we should also add that for some students, perseverance may pay off. Firstly, the students' standards for the evaluation of experiences of competence adjust over time, so students who did not feel successful because they could not solve their task sheet completely, may some weeks later feel that this is normal without any feeling of incompetence. Secondly, the individual reference norm with its benefits can only be used after some personal experiences. This becomes also visible in the different forms of positive experiences of competence, which build on previous negative experiences. This fits the observation of (Daskalogianni & Simpson, 2002), who found that many students have a "cooling off" period of motivation in the first weeks, which is followed by a "warming up" period.

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Interest and self-concept concerning two characters of mathematics: All the same, or different effects?

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Subject-specific interest and self-concept are theorized as important antecedents and results of learning processes. However, empirical results do not often support this hypothesis. In many cases, these affective variables are measured by questionnaires which refer to mathematics only in general terms. When transitioning to university mathematics however, the character of mathematics changes: from a school subject based on calculating and solving real-world problems to a scientific discipline. We surveyed individual interest and self-concepts concerning these two different characters of mathematics. First results of a longitudinal study with 331 university students indicates the power of this approach to provide a deeper insight into the role of learners' affective learning antecedents.

Theoretical background

Learning mathematics at university

The transition from school to university is experienced as an exciting and challenging period by many first-year students. Two differences between learning mathematics at school and at university are frequently described in the literature: a shift in the character of mathematics and different demands related to the learning cultures (Gueudet, 2008). In school, mathematics is mostly taught as a tool to solve problems arising in private and professional life. The special character of mathematics is often called "school mathematics", relating to a central role of technical calculations and modelling real-world problems. In contrast to that, mathematics as a scientific discipline is dominated by formally defined abstract concepts and deductive proofs at university (university mathematics) (Dörfler & McLone, 1986).

Interest and self-concept in learning processes

Interest and self-concept are frequently theorized as important affective antecedents and results of subject-specific learning processes. Both of them describe a certain kind of person-object-relationship (Marsh, Trautwein, Lüdtke, Köller, & Baumert, 2005). While interest relates to enjoyment, value, and an intrinsic curiosity connected to a subject by an individual, subject-related self-concept refers to the individual self-image of one person concerning his or her subject-related performance and skills.

Many authors argue that high interest and a positive self-concept are important for successful learning processes, also in university mathematics (c. f., Valle, Cabanach, Núnez, González-Pienda, Rodríguez, & Pineiro, 2003). However, empirical findings do not frequently support this hypothesis (Hailikari, Nevgi, & Komulainen, 2008). We suggest that this gap between theoretical assumptions and empirical results might trace back to the way these motivational components are measured. Research on the secondary level has unveiled that interest, as sur-veyed by questionnaires, is not at all a fixed construct, but that its structure

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undergoes sub-stantial changes over time (Frenzel, Pekrun, Dicke, & Goetz, 2012). Usually, the self-report scales used to measure interest and self-concept relate to mathematics in general terms, and not to a specific character of the discipline. Thus, various students might report interest and self-concepts relating to different characters of the discipline, leading to inconsistent results. In the SISMa project (Self-concept and Interest when Studying Mathematics), we study, among others, the hypothesis that affective characteristics differentiate between these characters of the discipline.

Measuring interest and self-concept in SISMa

In previous studies, data concerning mathematics-related interest and self-concept were mostly collected by questionnaire items which do not determine a specific character of the discipline (e.g., "Mathematics is very important for me" or "I am good in mathematics"). Nevertheless, the character of mathematics that is taught at university differs substantially from the one students experienced in school. This is why we develop measurement instruments in which we specifically address interest and self-concept concerning school mathematics and concerning university mathematics.

Table 1. Overview of measurement instruments for interest

	School mathematics	*University mathematics*
Institution	In school, mathematics was very important for me. (5 items)	I am interested in mathematics with which you deal at university. (5 items)
General practice	Calculating: It is exciting solving difficult equations. (6 items)	Proving: Reading mathematical proofs is fun. (4 items)
	Solving real-world problems (modelling): I find it interesting to solve real-world problems with mathematics. (6 items)	Using formal-symbolic representations: It is fun to exactly define mathematical concepts. (4 items)
Situated practice (e.g., It would be fun to deal with this task.)	Calculating: "Let f be $f(x)=\frac{\sqrt{1+x}\cdot e^x}{4+x^2}-1$. Calculate the extrema of the function f." (12 items)	Proving: "Let $f: IR -> IR$ be a differentiable function. Show that f is continuous." (12 items)
	Solving real-world problems (modelling): "By metal, you should produce a cylindrical can with a prescribed volume. For which radius is the material consumption minimal?" (12 items)	

Notes: Measuring on a four-point likert scale from 0: I don't agree to 3: I agree.

Apart from surveying interest relating to "mathematics in school" and "mathematics at university" in general, we took two different approaches to operationalize the different characters of the discipline. Firstly, we chose four different mathematical practices, two of which are predominantly important in school mathematics (performing complex calculations, solving real-world problems) and two are predominantly relevant for university mathematics (proving, using formal-symbolic representations). Secondly, to substantiate our measures with a more situated approach, we chose tasks which requested one of three of these practices (performing complex calculations, solving real-world problems, proving) in a prototypical way on the level of university mathematics (Rach, Heinze, & Ufer, 2014). We surveyed students' interest and self-concept relating to these practices in general resp. their interest and self-efficacy relating to the tasks. Table 1 gives an overview of the instruments.

After piloting these instruments, we used them f. e. to study if facets of interest and self-concept relating to different characters of mathematics can be differentiated in first-year university students. In this abstract, we concentrate on the interest measures.

Design

We conducted a longitudinal study with a sample of first-year students in two courses from five different study programs (n = 97 general mathematics, n = 96 financial mathematics, n = 99 teacher education for the high attaining school track, n = 16 teacher education for other school types, n = 18 computer science, n = 5 missing). We applied the same questionnaires on the first day of the semester (N_{T1} = 331) and after six weeks (N_{T2} = 199).

First results

Descriptive data and internal consistence of these scales

Table 2. Means (M), standard deviations (SD) and Cronbachs' alpha (α) for interest scales at start of semester ($T1$) and after six weeks ($T2$).

Scale	Start of semester (T1)		After six weeks (T2)	
	M (*SD*)	Cronbachs' α	*M* (*SD*)	Cronbachs' α
Mathematics	2.22 (0.43)	.74	2.06 (0.50)	.80
School mathematics	2.10 (0.58)	.77	2.16 (0.62)	.82
University mathematics	2.01 (0.60)	.87	1.71 (0.69)	.89
General practice: Calculating	2.20 (0.46)	.69	2.05 (0.47)	.73
General practice: Modelling	2.03 (0.58)	.81	1.93 (0.59)	.85
General practice: Proving	1.83 (0.62)	.76	1.51 (0.68)	.84
General practice: Using repre.	1.92 (0.59)	.71	1.81 (0.61)	.77
Situated practice: Calculating	2.27 (0.56)	.91	2.05 (0.58)	.89
Situated practice: Modelling	2.28 (0.53)	.90	2.06 (0.55)	.90
Situated practice: Proving	2.32 (0.54)	.92	2.12 (0.52)	.90

Notes: N_{T1} = 263 (university mathematics), N_{T1} = 314-331, N_{T2} = 186-199.

Cronbachs' alpha values indicate sufficient internal consistency of our newly developed scales. Exploratory factor analyses mostly replicated the theoretical structure of the instrument, with the exception that the facets relating to the practices "Proving" and "Using representations" collapsed into one factor in all analyses. So, we can combine these two highly correlated facets, see below, into one scale for further analyses.

Relationships between interest concerning different mathematical practices

As we argue, calculating and modelling are important activities in school mathematics where-as proving and using formal-symbolic representations are predominant in university mathe-matics. So we expected strong correlations only between individual interest on the first two and between the last two practices. Table 3 indicates that these assumptions were largely met.

Table 3. Correlations between general practices at the start of the semester **($T1$)**

	Modelling	Proving	Using representations
Calculating	.31 **	.16 **	.19 **
Modelling		-.07	.03
Proving			.59 **

Notes: N_{T1} = 331, ** p < .01.

Perspectives

Interest and self-concept are variables that indicate a person-object-relationship. As the "object", which is taught and learned, changes during the institutional transition from school to university, it is necessary to examine interest and self-concept concerning mathematics in a differentiated way. One explanation for problems in finding relations between interest in mathematics and learning gain in mathematics higher education possibly lies in this differen-tiation of the constructs: Students might just have "a different mathematics" in mind when answering a general interest questionnaire on the first day of the semester as the mathematics they will encounter in their subsequent learning processes. The selected results mention that the developed instruments can be used to differentiate between interest and self-concept con-cerning various characters of mathematics. Thus, we will analyze the role of affective compo-nents when learning mathematics during the first semester with our instruments in the future.

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6. LEARNING AND TEACHING OF SPECIFIC MATHEMATICAL CONCEPTS AND METHODS

Understanding and advancing undergraduate mathematics instructors' mathematical and pedagogical content knowledge

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Mathematics PhD students and instructors are not being supported in acquiring coherent and rich meanings for foundational ideas that surface in some undergraduate courses. This leaves future mathematicians unprepared to connect more advanced mathematics topics to foundational ideas, making undergraduate students' mathematics learning experiences less meaningful. In this session I will share select data from a study that explored the mathematical meanings that mathematics PhD students and mathematics instructors hold about ideas of average rate of change and exponential growth. These findings raise questions about the knowledge that is guiding mathematicians' curricular and instructional choices.

In recent years researchers have investigated the pedagogical and mathematical content knowledge for teaching ideas in both elementary (e.g., Hill, Ball & Shilling, 2008) and secondary (e.g., Silverman & Thompson, 2008; Tallman, 2015) mathematics. Other researchers have investigated and described key components of an inquiry orientation to teaching undergraduate mathematics (Rasmussen & Kwon, 2007). These studies raise questions as to whether undergraduate mathematics instruction might benefit by engaging mathematics instructors in explorations of what is involved in understanding, learning, and teaching key ideas of courses they teach.

The content focus of undergraduate mathematics curriculum and instruction within a mathematics department is commonly determined by the department's mathematics faculty, most of whom have had few opportunities to consider what is involved in understanding and learning key ideas of the courses they teach. Though these faculty successfully completed graduate courses in mathematics, studies show that completion of more mathematics coursework does not necessarily improve a teacher's instructional practices or understanding of fundamental ideas they teach (Speer; 2008; Speer, Gutmann, & Murphy, 2005). Shifting undergraduate instruction to be more conceptually oriented will require interventions that address what is involved in understanding and learning a course's key ideas (Thompson, Carlson, & Silverman, 2007), with sustained interventions that lead to rich connections among ideas. An instructor with strong understandings and connections is more able to engage students in meaningful and coherent instruction (Tallman, 2015).

This study is situated in the context of an intervention to support graduate students in constructing conceptual structures that will enable them to be highly effective teachers[1]. We

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probed graduate students' meanings for topics that are foundational to learning calculus and topics central to the current course they were or currently are teaching. We share our analyses of the graduate students' meanings relative to *average rate of change* (AROC) before and after the intervention. We also highlight graduate students' difficulties in communicating about *exponential growth* (EG) as a second example that illuminates the need for content-focused professional development of future mathematics faculty.

An individual constructs meanings through repeated experiences and reflection on those experiences. Though meanings exist in the mind of the individual, we can only construct models of *expressed meanings*. We do this by analyzing the individual's written and oral communications about mathematical ideas. We offer conceptual analyses of productive meanings for AROC and EG against which we compare the conveyed meanings of the graduate students in our study.

An individual constructs a productive meaning for *average rate of change* when he conceptualizes a hypothetical relationship between two varying quantities in a dynamic situation (Thompson, 1994). Namely, given covarying quantities A and B, and a fixed interval of measure of quantity A, the AROC of quantity B with respect to quantity A is the constant rate of change (CROC) that yields the same change in quantity B as the original relationship over the given interval. In order to understand this idea meaningfully, an individual must first conceptualize two quantities changing together and then apply the idea of CROC to approximate the rate of change of the two covarying quantities on a specified interval.

Since exponential growth is sometimes expressed as a percent increase or decrease of some amount that is repeatedly iterated for some specified interval of the independent quantity, a productive meaning for exponential growth includes the ability to represent iterations of a percent change over multiple intervals of the independent quantity. An individual may conceptualize equal changes in the input quantity corresponding to consistent multiplicative changes in the output quantity. The factor by which the output quantity increases for each *1-unit* interval of the independent quantity is called the *growth factor*. An individual's meaning is more robust if he can imagine an n-unit growth factor as the scale factor for the output quantity for *any* n-unit change in the input quantity and all possible values of n (i.e., n is a positive real number). The individual must have strong, foundational meanings for quantity, multiplicative comparisons, and exponentiation in order to construct a well-connected scheme for EG.

The goal of the intervention was to prepare future mathematics faculty to teach effectively by supporting them in developing rich and connected mathematical meanings of central ideas they are teaching in undergraduate precalculus and calculus. Participants respond to problems designed to perturb their meanings and improve their meanings during an initial 2- or 3-day workshop. Workshop leaders repeatedly asked participants to provide a conceptually oriented explanation of their solutions, including explanations that reference quantities

[1] We define highly effective teachers as ones who: i) exhibit strong mathematical conceptions in their interactions with students; ii) attempt to understand their students' thinking; iii) make instructional moves and pose questions for the purpose of supporting students in constructing meaningful conceptions and connections.

and justification for their approach. During the semester when teaching they attend weekly 90-minute seminars concurrent with using a research-based curriculum.

We gathered data from 84 mathematics graduate students or instructors at three public, PhD-granting universities within the United States. The graduate students' teaching experience varied between 0 and 11 years at the K-12 and tertiary levels. Participants responded to mathematical tasks and hypothetical teaching scenarios both in writing and during semi-structured clinical interviews. Portions of the workshops were recorded. Members of the research team analyzed videos and written data through an iterative process of identifying and refining themes relative to our conception of a productive expressed meaning for AROC and EG (Strauss & Corbin, 1990).

Participants' meanings for AROC before experiencing the intervention predominantly fell into two categories: computational (e.g., "delta y divided by delta x") or geometric (e.g., "steepness of a graph"). More interesting to note, however, is that these meanings were extremely resistant to change, as was the graduate students' fluency in speaking about AROC using quantitative descriptors (e.g. "as x <u>increases</u>" instead of "as x <u>goes</u>"). Table 1 contains sample responses that demonstrate the various levels of fluency in describing AROC at various stages of the intervention: from pre-intervention to mid-workshop to post-teaching AROC.

Pre-Workshop	Day 2 of Workshop	Post-Teaching AROC to Class
• A straight line between two points on a graph. • As one variable changes for every unit, how much is the other variable changing. Slope. • Steepness of a graph, like how steep or how flat it is. • It's the predictive effect of changing one variable and the amount and how it's going to affect the other variable. One quantity affecting change in another quantity.	• The AROC is the secant vector between two points on a graph connected by some function. • The CROC one would go to achieve the same change of function in question. • The CROC needed to achieve the same change in the y-value over the same change in the x-value. • A way of describing a change in your y-value in terms of CROC of x-value.	• The CROC to have traveled the same amount distance in the same amount of time. • The CROC that you have to apply to your variable to get a fixed value of your answer. • The CROC required to cover the same change in the dependent variable over the same change in the independent variable. • Sum all the rate of change, then divide by the total number.

Table 1: Responses to "What is the meaning of AROC?"

Early on, geometric interpretations range from true but limited ("slope") to strictly false (AROC is "a straight line"). Mid-workshop, participants struggled to move away from thinking about the geometric object they visualize ("the secant vector between two points") to which property of that object is described by AROC. Post-teaching, most participants correctly communicate that AROC is the CROC needed to satisfy some condition; however, the verbiage used in describing that condition often entails a sense of motion ("CROC required to *cover*...") or is restricted to describing the special case of average speed ("CROC to have traveled..."). Surprisingly, two of the graduate students post-intervention persisted in making a common student error of confounding AROC with arithmetic mean ("Sum all the rate[s] of change…").

A student was asked to write a function g to determine the number of bacteria in a culture for any number of seconds that have elapsed, as elapsed time increases continuously. She was told that the number of bacteria in the culture increased by 23% every 12 seconds. The student defined x to represent the number of seconds the culture has been growing. What answer do you want your students to provide and what thinking do you expect students to engage in to produce a correct answer?

Figure 1: Exponential Growth Task

Similarly, participants struggled significantly pre- and post-intervention to discuss the idea of exponential growth. In particular, participants relied heavily on procedural knowledge to solve tasks and had difficulty explaining the meaning behind the symbols or to offer conceptual explanations. On the task in Figure 1, over 30% of graduate students did not produce a correct response prior to participating in the intervention. Many of those who produced a correct answer were unable to explain why their solution was correct. Table 2 shows the distribution of responses from pre-intervention graduate students on this written task.

Response Properties	Sample Response	Response Count
Valid formula and notation	$g(x) = n\,(1.23)^{\ t/12}$, where A is initial amount of bacteria and x is the number of elapsed seconds	14
Valid formula but other issues	$g = n\,(1.23)^{\ t/12}$	13
Invalid formula	$g(x) = Ae^{\,0.23x}$	5
Did not provide formula	I would help [students] think how much increased by one second and how many seconds it needs to increase by 1%.	7
	Total	39

Table 2: Pre-intervention responses to task in Figure 1

Participants with at least one semester of teaching experience were interviewed and given the same exponential growth task framed in the context of a teaching scenario. All interviewees gave procedural responses, though several participants reflected on this and expressed discontent.

Our analyses point to weaknesses in future faculty's meanings for foundational concepts. As an instructor's meanings impact the nature of his teaching practices and the scope of what he is able to do with his students, it is necessary to address these gaps among future faculty to make mathematically rich and meaningful experiences possible for students.

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Linking elementary notions of limit concepts

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Limit of a sequence, limit of a function at a point and limit of a function at infinity are often introduced at different times in the undergraduate curriculum, can have different definitions and yet have near identical symbols. In many cases, few explicit connections between them are revealed by the teacher, and existing research either treats them separately or fails to distinguish between them. Our study examined what, if any, links are made by students exposed to stimuli designed to represent these notions. Participants engaged with two different types of card sorting task to expose the personal categorizations and connections between these different types of limit. Results suggest that few participants made connections that fit an overarching notion of limit.

The problem of limit(s)

Oehrtmann (2008) neatly encapsulates some of the problems which students encounter with the teaching of limits. He notes that one of the probable rationales for the teaching of limits is the exposure to formal definitions and proofs, perhaps in a rigorous treatment of the development of calculus – presumably as an example of the definition-theorem-proof approach to developing much university level pure mathematics. He also considers a second rationale as developing an intuitive understanding of the limit concept, but recognizes that the problems with doing so (and with co-ordinating the intuitive with the rigorous) leads many to de-emphasizing limits in developing calculus.

Much of the existing literature points to a key problem in understanding and co-ordinating a dynamic notion of a limiting process with a static limit (e.g. Sierpińska, 1987; Cottrill et al., 1996). Some even argue that limit is *necessarily* thought of as co-ordinated dynamic sequences (Lakoff and Nüñez, 2000) seemingly unaware of infinitesimal approaches which both are mathematically rigorous and, at least for some, can be psychologically stable (Ely, 2010; Borovik and Katz, 2012).

However, one area which is often ignored is that elementary limit notions can be introduced three times, potentially in three different ways, but with near identical compound symbols: limit of a sequence at a point ($\lim_{n\to\infty} a_n$), limit of a function at a point ($\lim_{x\to a} f(x)$) and limit of a function at infinity ($\lim_{x\to+\infty} f(x)$). Existing research has tended to treat these notions separately (McDonald, Mathews, and Strobel, 2000 vs. Güçler, 2013 vs. Kidron, 2011) or to examine two or more variants in the same study without discussion of their connections (e.g. Tall and Vinner, 1981; Elia, Gagatsis, Panaoura, Zachariades, and Zoulinaki, 2009).

In the usual UK university calculus and analysis curriculum, at least, these three notions are introduced separately. In some cases (e.g Bryant, 1990) limits of functions at a point are

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defined in terms of limits of sequences, but in other cases (e.g. Spivak, 2006) each elementary limit notion is defined separately and with no reference to the others.

The study reported here investigated the extent to which students who had experienced this kind of disconnected limit curriculum did, or did not, make links between these different limit notions. Constraints of space mean that we only briefly outline the methods and give illustrations of responses before explaining the categories of response seen in the data.

Methods and Participants

Given that the aim of the study was to uncover potentially implicit links students may have between concepts that have been presented separately, the methods needed to provide a mechanism for participants to bring those connections to the fore spontaneously and to give them a stimulus to discuss them. We thus developed a card sorting activity in which participants were presented with 20 cards (see screenshot in figure 1) which were designed to draw attention to elementary limit concepts, generally using expressions the participants could be expected to understand.

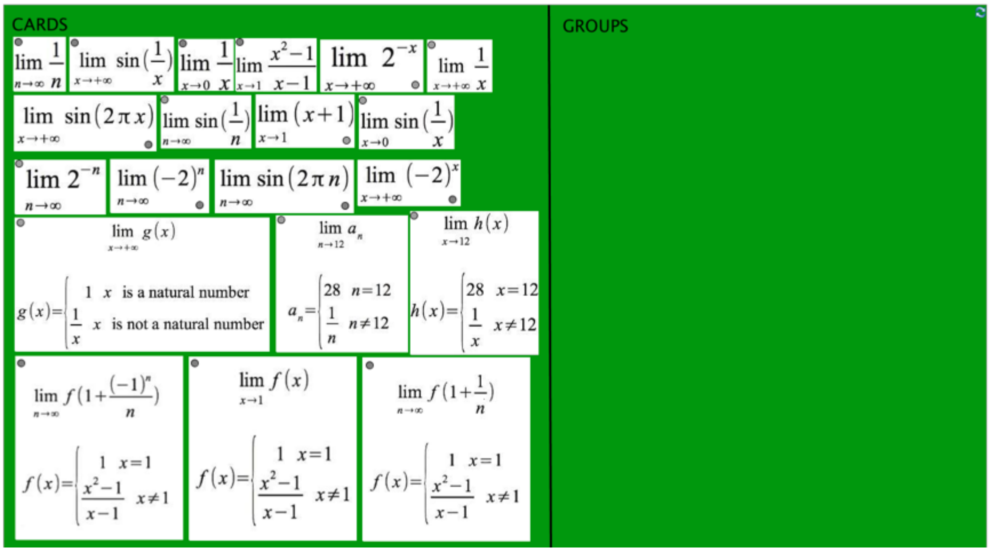

Figure 1: The Card Sorting Activity stimuli set

Students in their second year of their degree at a research-intensive UK university were recruited to take part in the study. Fourteen participants worked one-to-one with a researcher on the tasks, for which they were allowed as much time as they wished. In fact, the tasks took between forty minutes and one hour. To aid recording, the card stimuli were represented on a computer screen as click-and-drag icons and the sessions were videoed (as piloting showed that information was lost unless we could record participants' gesturing to cards when they referenced them).

The participants were asked to complete two tasks – the first (and the only one reported here) was open sorting and the second was to talk about pairs or groups of cards highlighted by the researcher. For the open sorting task, they were asked to sort the cards according to any criterion they liked and were encouraged to do this as many times using as many different criteria as they could.

Illustrative Results

The most common first response to the sorting task was to categorize according to the core expression within the compound symbol, ignoring the limit aspects of each card. For example:

S9: *These are all the same because they all involve the sine function.*

S3: *They've got constants to some power ... then here we've got reciprocal functions.*

However, as participants went through cycles of categorizing, criteria which we might see as more or less analytic appeared. The distinction between limits of functions and limits of sequences was often early to emerge. In many cases, this was because participants attempted to classify according to the variable names in the compound symbol. Note that, in the design of the task, no attempt was made to specify the reference set for n and x, but every participant took the convention that expressions involving n were sequences and those involving x were (real) functions.

S14: *... to begin with ... separate the 'n's and 'x' ... there are so many obvious differences between the cards, this is one of them.*

The final 'surface' feature which was used was the limit point – we categorise this as 'surface' as it appears as though the criterion was determined as simply as being the symbol appearing after the arrow on each of the cards

S11: *We've got limits as 'x' or 'n' approaches infinity in one group ... as 'x' approaches zero ... as 'x' approaches come constant.*

As students focused more on the limit properties of the expressions, it became clear that many did not make appropriate connections or distinctions between types of limit. For example, in considering the cards $\lim_{x\to+\infty} sin(2\pi x)$ and $\lim_{n\to\infty} sin(2\pi n)$ some students saw them as interchangeable

S8: *It looks different because that's '$+\infty$' and that's '∞'. But I think that [∞] implies positive infinity, so they're the same.... It's the same thing, we could just say "let 'x' equal 'n' " and then it becomes the same.*

However, others had more connection:

S7: *This [$lim_{n\to\infty} sin(2\pi n)$] is a subsequence of this [$lim_{x\to+\infty} sin(2\pi x)$].*

There was also some emerging use of implicit notions of neighbourhood when students were comparing cards:

S11: [*Looking at* $lim_{x\to+\infty} sin\left(\frac{1}{x}\right)$ *and* $lim_{x\to 0} sin\left(\frac{1}{x}\right)$]

So, when that tends to infinity. I know you shouldn't split it up but, $\frac{1}{x}$ has a limit as x tends to infinity, which we know is going to be zero and we know that sine is defined at zero, whereas sine isn't defined at $\frac{1}{0}$, which is undefined. So we can always say that this has a limit, because we know what sine is defined at, at zero, zero, whereas sine doesn't have a definition as, as x gets very small, so if we say that x isn't equal to zero, x is very very small,

$sin(x)$ could be anything between plus and minus one because it's an oscillating function.

Discussion

In such a short space we cannot illustrate the full richness of the students' responses, but we found that, after an initial focus on surface features of the compound limit symbol, students were able to make some links across elementary limit notions. However, these did not always fit the formal mathematics: for example functions of x as x tends to infinity were simply equated with limits of functions in $1/x$ as x tends to zero and some equated the limit of a function at a point with the function evaluated at the point (despite being second year students with three full courses in analysis including notions of continuity behind them). There was only very occasional reference to a neighbourhood notion.

We believe the full analysis of our study shows that teachers might give more attention to the issue of working with a compound symbol and, particularly with making connections between the elementary limit concepts. Moreover, by discussing elementary limit concepts together and explicitly drawing out the idea that each is an example of neighbourhoods mapping to neighbourhoods (with some suitable interpretation of neighbourhood), teachers may help decrease the problems Oehrtmann (2008) saw in connecting the rigorous with the intuitive in the context of limit.

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What level of understanding of the derivative do students of economics have when entering university? – Results of a pretest covering important aspects of the derivative

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The concept of the derivative plays a major role in economics, for example in cost theory. A proper understanding of the concept and its aspects is therefore necessary for being able to deal with the concept in economics in a reflective manner. To get an idea which level of conceptual understanding first-year students of economics have at the beginning of their studies, a pretest on the derivative was submitted to beginner students of economics at the University of Paderborn. Some results of the pretest and possible consequences for teaching are presented here.

Background

The concept of the derivative is very important for economics and has many applications there, for example in production and cost theory. Therefore, an appropriate understanding of the concept and its applications in economics is essential for students of economics. The study presented here is part of a larger research project "Understanding of the derivative by students of economics" (my PhD-Thesis, supervisor: Prof. Dr. Rolf Biehler), in which the following research questions are addressed:

1) Which understanding of the concept of derivative do students of economics need to have?
2) Which understanding of the concept of derivative do students of economics have before attending any mathematical course at university?
3) Which understanding of derivative do students of economics have after the math course, especially concerning its use in economics in the example of marginal cost?

The study presented here is a pretest, administered to students of economics at the university before their math course, with the aim to find out what pre-knowledge concerning the concept of the derivative the students have when entering university (Question 2).

Theoretical Framework

According to Zandieh (2000) the concept of derivative is connected with three other mathematical concepts that she calls layers of the derivative:

1. The concept of ratio/rate (relevant for understanding the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$ as he first step for getting from a function f to its derivative)

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2. The concept of limit (relevant when taking the limiting process $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$)
3. The concept of function (relevant for the transition from a single value of the derivative $f'(x_0)$ to the derivative function f')

Each of the concepts can be seen as a process-object pair. For the layer of the limit for example, the process is the limiting process, and the object is the value of the limit. Furthermore Zandieh (2000) mentions multiple representations for the derivative that students ought to know: a) graphically as the slope of the tangent line at a point, b) verbally as the instantaneous rate of change, c) physically as speed or velocity, or d) symbolically as the limit of the difference quotient.

For students of economics an additional aspect of major importance has to be added: the economic interpretation of the derivative, which is not represented in the framework. In economics, the derivative is often interpreted as the absolute change of the values of the function when the independent variable increases or decreases by one unit. In case of a cost function K with the independent variable x representing the output, for example, the derivative $K'(x)$ is often interpreted as the additional cost while increasing the production from x to $x+1$ units. This interpretation of the derivative is very practical in economics because economic functions often have discrete units, but it differs from the view on the derivative as local rate of change or as slope that students learned at school. No rate and no limit are involved in the economic interpretation. The connection between the mathematical concept of the derivative and the economic interpretation, however, is often not made clear in books of economics. To understand the interpretation properly the limiting process behind the derivative has to be deencapsulated (Asiala et al., 2000) and the approximation aspect of the limit has to be used when approximating the difference quotient with the derivative and vice versa (Çetin, 2009; Williams, 1991). Nevertheless, knowing the derivative as a rate is still necessary to explain the unit of $K'(x)$, which is Euro per unit (if the cost $K(x)$ for an output x is given in Euro) and not Euro like the economic interpretation would suggest.

Structure of the test

The pretest consists of seven subtests. Most of the subtests cover one of the aspects of the derivative mentioned in the above framework by Zandieh (2000) (process-object layers and representations). In addition, two further subtests were added: one containing tasks to differentiate functions because of the assumption that students of economics also lack of this procedural knowledge, and one part concerning the economic interpretation of the derivative and related mathematical background knowledge (use of the derivative for approximation).

The seven subtests were the following:

1. The aspects of the slope of a linear function (6 items)

   This subtest contains tasks about the students' understanding of the slope of a linear function based on the different aspects of slope mentioned by Nagle, Moore-Russo, Viglietti, & Martin (2013)

2. Understanding of the difference quotient (7 items)

Goal of this subtest was to see if the beginner students know at least one interpretation of the difference quotient (e.g. as average rate of change, as growth factor, or as mean of equidistant growths (Malle, 1999)). It also contains questions addressing misconceptions of the difference quotient like interpreting it as the absolute change or the arithmetic mean (Carlson, Oehrtman, & Engelke, 2010; Thompson, 1994).

3. The interpretation of the derivative as rate of change in a context (10 items)

   In this subtest the students have to solve tasks involving the interpretation of the derivative in a context and requiring covariational reasoning, for example tasks involving graphs of filling processes (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002)

4. The geometric interpretation of the derivative as slope of the tangent line (11 items)

   This is a very important aspect of the derivative and must be known to the students. The tasks of that subtest mainly focus on the slope/height-confusion (Leinhardt, Zaslavsky, & Stein, 1990).

5. The analytical tangent line (8 items)

   This subtest covers aspects and misconceptions of the analytical concept of the tangent line. The misconceptions mainly occur because of no extension of the concept of the tangent line at a circle (Biza, 2007; Vinner, 1982). A proper understanding of the tangent line is important to understand the geometric interpretation of the derivative.

6. The algebra of the derivative (6 items)

   In this subtest the ability of differentiating elementary functions by applying certain differentiation rules is tested.

7. The economic interpretation of the derivative and directly related background knowledge (5 items)

   This subtest covers a possible interpretation of the derivative in economics. It was included to find out if the students know about the economic interpretation as the absolute change of the values of the function while increasing the production by one unit that is used in many books of economics, e.g. Pindyck and Rubinfeld (2009) or Wöhe and Döring (2002), or if they can intuitively give another suitable economic interpretation of the derivative.

   It also contains tasks involving the usage of the derivative for the approximation of values of a function (underlying idea: the derivative $f'(x)$ as the slope of the linear function that is the best linear approximation of the function f in a neighborhood of x (Blum & Kirsch, 1979; Danckwerts & Vogel, 2006), which is important to understand why the derivative $K'(x)$ of a cost function K (x the output of a product) is an approximation of the additional cost $K(x+1) - K(x)$.

Altogether the test consists of 53 items. 38 items are multiple-choice questions, each having exactly one correct answer, 15 items are open-ended questions. Except for two open-ended questions, for which 2 points were given in total with the option of partial credit of 1 point, each correct answer yielded one point. Therefore a maximum of 55 points could be reached.

Data Collection

The test was administered to students of economics at the University of Paderborn in September 2015 in the bridging-course before any math-course (N = 143). The topics of the test had also not been covered in the bridging course yet. The duration of the test was 45 minutes. No calculators were allowed in the test.

Data Analysis

The data was analyzed with classical test-theory. The Cronbach's alphas of the subtests were mostly good (table 1).

Subtest	**Alpha**
The aspects of the slope of a linear function	0.703
Understanding of the difference quotient	0.593
The interpretation of the derivative as rate of change	0.706
The geometric interpretation of the derivative as slope of the tangent line	0.755
The analytical tangent line	0.688
Algebra of the derivative	0.708
The economic interpretation of the derivative and directly related background knowledge	-

Table 1: Values for Cronbach's alpha of the subtests of the pretest concerning the understanding of the derivative by students of economics

For the subtest "Economic interpretation of the derivative and directly related background knowledge" no value for Cronbach's alpha is given in table 1 because this subtest is heterogeneous and not one-dimensional and therefore, Cronbach's alpha is not a reasonable estimator for the reliability of this subtest.

Results

On average the freshmen students of economics received 25.44 out of the maximal 55 points, a little bit less than 50% (median: 26 points). The standard deviation with 7.96 points is pretty high, which means a very high heterogeneity in the pre-knowledge concerning the understanding of the derivative among the students. The distribution of the achieved points is shown in figure 1.

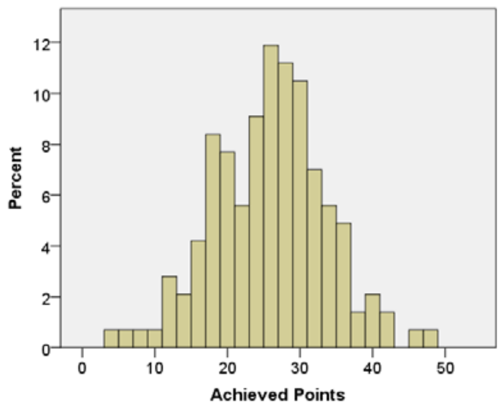

Figure 1: Distribution of the achieved points in the pretest (N = 143)

The pre-knowledge concerning the derivative is not only heterogeneous among the students but also among different aspects of the derivative represented by the subtests of the pretest. The normalized means of the subtests can be seen in table 2.

Subtest	**Normalized Mean**
The aspects of the slope of a linear function	0.61
Understanding of the difference quotient	0.16
The interpretation of the derivative as rate of change	0.57
The geometric interpretation of the derivative as slope of the tangent line	0.56
The analytical tangent line	0.57
Algebra of the derivative	0.40
The economic interpretation of the derivative and directly related background knowledge	0.22

Table 2: Normalized means of the subtests (the normalized mean of a subtest is the mean of the achieved points divided by the maximal number of points attainable in the subtest)

From the means it can be seen that the students' pre-knowledge concerning the geometric interpretation of the derivative and the interpretation as rate of change is not bad. Many of the students are able to interpret the derivative as slope correctly. The majority of the students do not have the slope/height confusion (Leinhardt et al., 1990) as can be seen from the results of two sample items from the test (figure 3).

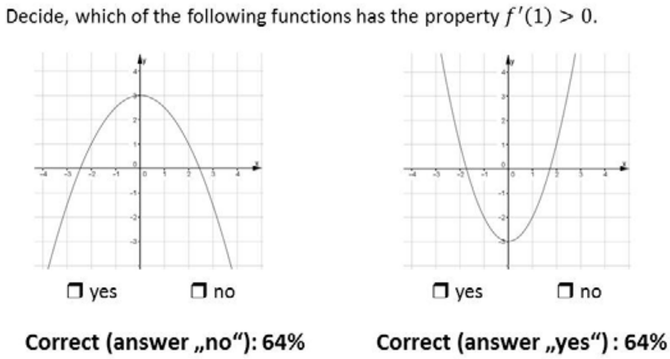

Figure 2: Percentage of correct answers for two items concerning the geometric interpretation of the derivative (N = 143)

As can be also seen in table 2 many students are able to deal with the concept of the derivative in contexts involving the interpretation as local rate of change and covariational reasoning. As an example the results on a tasks about filling graphs (based on the bottle problem from Carlson et al. (2002)) are shown in figure 3.

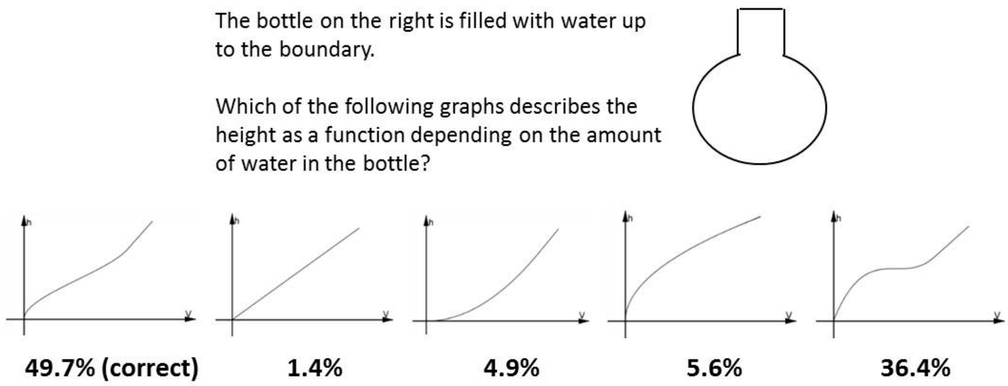

Figure 3: Students' answers on a task concerning rate of change and covariational reasoning (N = 143)

If the detail that the inflection point does not have slope zero is not taken into account even 76.6% of the students were able to choose a suitable filling graph (answers 1 and 5).

However, most of the students do not have an understanding of the difference quotient (see table 2). But this understanding is very important for being able to understand the derivative as the limit of the difference quotient that is necessary for understanding the connection between the derivative as a mathematical concept and its economic interpretation, like it was presented above (page 2). The results of one sample item can be seen in figure 4.

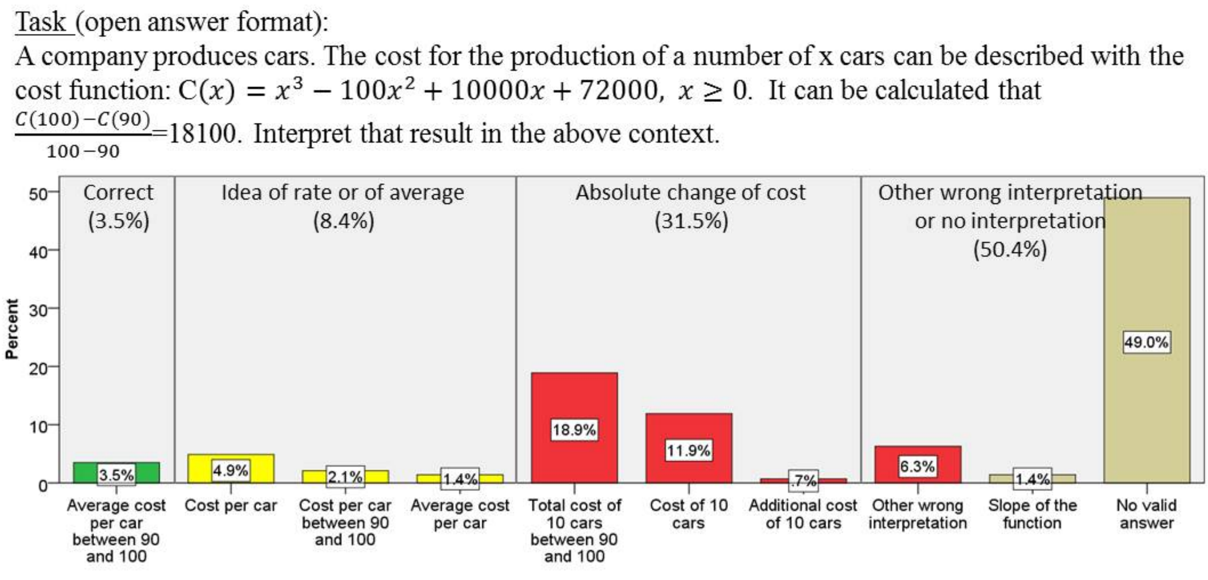

Figure 4: Categorized answers on a task concerning the interpretation of the difference quotient in a context (N = 143)

Another problem for students of economics is the calculation of derivatives (see table 2). The results for the six functions of the subtest "Algebra of the derivative" can be seen in table 3.

Function	Correct answers for the derivative (in%)
$f(x) = x^3 + 3x^2 - x + 2$	77.6%
$f(x) = \sqrt{x} + 3x$	39.2%
$f(x) = e^{2x}$	38.5%
$f(x) = e^{x^2}$	32.2%
$f(x) = \frac{1}{x}$	31.5%
$f(x) = xe^x$	11.2%

Table 3: Results for the tasks to differentiate functions from the subtest "Algebra of the derivative" (N = 143)

It can be clearly seen that if the function is not a polynomial, only a minority of students is able to differentiate it correctly. Especially the low percentage of correct answers concerning the product rule is problematic because the product and the quotient rule are important for students of economics when changing between cost and average cost (average cost for an output x is the cost for the output x divided by the output x itself).

A last important result from the pretest is related to the economic interpretation of the derivative. The interpretation of $K'(x)$ of a cost function K (x is the output of a product) is neither known by the students from school nor is it as intuitive that the students would state it spontaneously. This can be clearly seen from the results of task, in which the students were asked to give an interpretation for the derivative in an economic context (figure 5).

Task (open answer format):
A company produces pens. The cost (in Euro) for the production of a number of x pens can be described with a cost function with the equation:
$C(x) = \frac{1}{30000}x^3 - \frac{1}{100}x^2 + 2x,\ x \geq 0$
It can be determined that $C'(200) = 2$. Interpret that result in the above context

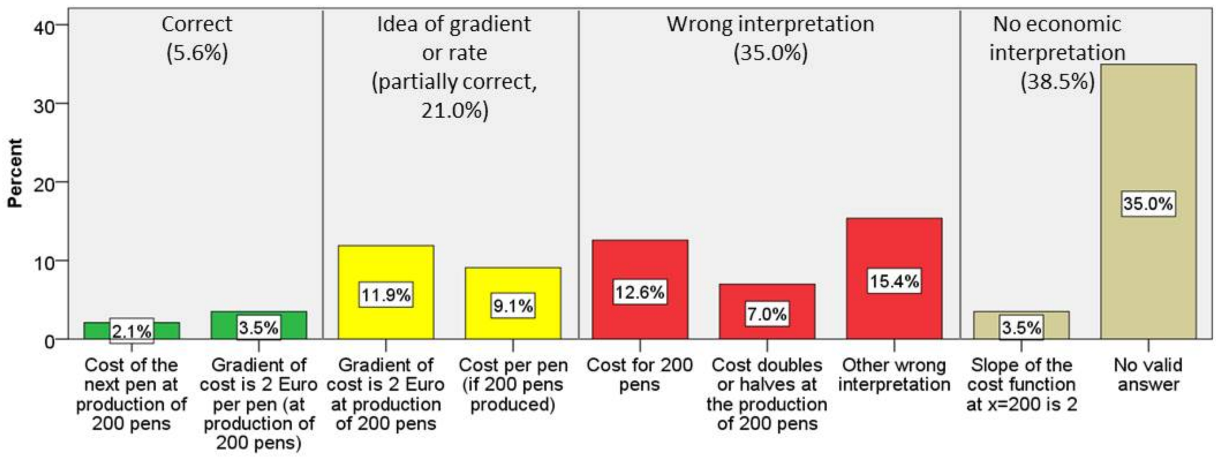

Figure 5: Categorized answers on a task concerning economic interpretation of the derivative (N = 143)

The first column represents the economic representation, which obviously very seldom mentioned. Instead the students seem to try to interpret the derivative with the idea of slope (third column) or of rate (fourth column). In addition, misconceptions like the slope/height-confusion can be seen.

Consequences for teaching and prospects for further research

One main result of the study is that the pre-knowledge concerning the concept of the derivative is very heterogeneous among the students of economics. However on average the beginner students bring a pretty solid knowledge base concerning the geometric interpretation of the derivative and the interpretation of the derivative as rate of change. However, the understanding of the difference quotient is very poor, which is very important for being able to understand the economic interpretation of the derivative properly (see second page of this paper). Therefore, a stronger emphasis has to be put on this in the lecture. Furthermore the students' procedural knowledge in differentiation is also not satisfactory. So differentiating cannot be taken as a prerequisite and has to be practiced in the math course. Concerning the economic interpretation of the derivative the pretest clearly shows that the interpretation is not known from school and that the interpretation does not evolve intuitively from the students' pre-knowledge. Therefore, the economic interpretation has to be introduced carefully in the course and has to be connected carefully to the students' previous knowledge and the previous understanding of the mathematical concept of derivative.

Concerning further research in the PhD-Project "Understanding of the derivative by students of economics" the result concerning the students' difficulties to use their pre-knowledge to give an adequate interpretation for the derivative in economics clearly leads to the question to what extent the students' are able to understand the common economic interpretation of the derivative $K'(x)$ of a cost function K (x is the output) as additional cost $K(x+1)-K(x)$, while increasing the production from x units by one unit after the math course at university (Research question 3 of my PhD-Thesis, see page 1 of this paper).

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Inquiry oriented instruction in abstract algebra

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Hans Freudenthal (1973) argued that that "groups are important because they arise from structures as automorphisms of those structures." (p. 109). This perspective was presented as an alternative to the traditional instruction treatment in which one starts by defining group and then proceeds formally and deductively proving theorems that follow from this definition. Burn (1996), following Freudenthal, argued that permutations and symmetries (and not the formal definition of group) should be regarded as the fundamental concepts of group theory. The purpose of this talk will be to share some results from a series of designed-based research projects focused on developing a course in introductory group theory that takes as its starting point the symmetries of geometric shapes.

Introduction

The Teaching Abstract Algebra for Understanding project (Larsen, Johnson, & Weber, 2013) is an ongoing effort to design and "scale up" an inquiry oriented group theory course. The design of the course was guided by the instructional design theory of Realistic Mathematics Education (RME). The course features a core of three instructional sequences, each focused on a fundamental concept. The point of departure for the course is an investigation of the symmetries of an equilateral triangle that features the guided reinvention of the group concept. The second reinvention sequence results in a definition of isomorphism as a formalization of the intuitive idea that if one simply changes all the names of the elements of a group one does not change the group. Finally, the quotient group concept is reinvented beginning with the intuitive notion of parity (even/odd) applied to the group of symmetries of a square. I will briefly describe each of these reinvention sequences. The instructional approach is inquiry-oriented in the sense that the students engage in authentic mathematical inquiry and the instructor must engage in inquiry into the students' thinking in order to support their mathematical activity. With this in mind, I will highlight key insights we gained while working with students and how these fed into the instructional design of each sequence.

Reinventing the group concept

The reinvention of group begins in the context of the symmetries of a geometric figure (Larsen, 2013). Students identify, describe, and symbolize the set of symmetries of an equilateral triangle. The group structure begins to emerge as a *model-of* the students' mathematical activity (Gravemeijer, 1999) as they begin to analyze combinations of pairs of symmetries. The transition of the group concept to a *model-for* more formal mathematical activity (Gravemeijer, 1999) is a long-term process that unfolds throughout the course. The first substantial step is a shift to working algebraically with the symmetries. The first key insights we gained from our own inquiry into students' activity were related to this shift.

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Key insights: developing a system of computational rules

In our first design experiment, the students happened to develop a set of symbols for the symmetries of a triangle that included composite symbols (e.g., FCL represented a flip across the vertical axis of symmetry followed by a clockwise rotation of 120 degrees.) When they used these composite symbols and tried to analyze a combination of two symmetries, they wrote down expressions like $F\,(FCL)$ that begged to be calculated using associativity, inverses, and the identity property: $F\,(FCL) = (F\,F)\;CL = N\;CL = CL$. Going into this first design study, we were unsure how the group axioms could arise from the students' mathematical activity because we expected the students to "compute" combinations by manipulating a physical triangle we provided. However, the pair of students we worked with spontaneously transitioned to paper-and-pencil calculations that anticipated the group axioms. Thus we learned two things from working with them and analyzing their activity. First, was that this activity of performing rule-based calculations was something that could be leveraged to develop the formal group concept. Second, was that this activity was promoted by their use of composite symbols. Drawing on this insight, the instructional design now includes tasks that promote the use of composite symbols and then engage the students explicitly in creating a rule-based computational system. This system of rules then becomes the subject matter for further (vertical) mathematizing, eventually leading to a definition of group.

Reinventing the isomorphism concept

The design of the isomorphism reinvention sequence began with a major misstep (Larsen, 2009). Consistent with the approach of "theory guided bricolage" (Gravemeijer, 1998), we attempted to implement a recommendation of Thrash and Walls, (1991). Thrash and Walls note that, because in a group table each element must appear exactly once in each row/column, there are only four ways to complete an operation table for a group of four elements. They observe that students can rename elements and rearrange the tables to verify that three of these tables represent isomorphic groups, and that by engaging in this activity they can understand isomorphism right away. The idea of initiating the reinvention of isomorphism by having students determine the number of groups of order four makes sense in light of the RME heuristic of *didactic phenomenology* (Gravemeijer and Terwel, 2000) because this problem presents a phenomenon that can be organized using the concept of isomorphism. Put simply, this is the kind of problem one needs isomorphism to solve, so it makes sense as a starting point. However, in our first design experiment, we found this staring point to be disastrous and our analysis of the students' activity revealed that there were important ways in which such a starting point does *not* align with the principles of RME.

Key insight: abstractly represented groups were not experientially real

The first pair of students we worked with could make no sense of the question of whether two groups (abstractly represented by 4x4 tables featuring the symbols A, B, C, and D) were "the same". And they could not determine what kinds of manipulation of these symbols and tables would change the group significantly and which would not. For example, they argued that since all of the groups used the same set they must all be different because the operation tables were different. We realized during our retrospective analysis, that these rather abstract tables did not represent groups to the students. In particular, the tables did not represent any system that was real to them in the way that the triangle symmetry group existed as a real thing that was represented by the symbolic system they developed. As a result of this insight, we redesigned the starting point, this time engaging the students in the task of determining whether a given "mystery table" could represent the symmetries of an equilateral triangle. This resulted in the students meaningfully manipulating symbols in a way that anticipated the isomorphism concept. In fact the procedure students use to address this question readily becomes new subject matter to be further mathematized eventually resulting in a definition of isomorphism.

Reinventing the quotient group concept

The reinvention of the quotient group concepts begins with the students considering the parity of the integers (Larsen & Lockwood, 2013). This was a starting point advocated by Burn (1996) as an easily understood example of a quotient group. Dubinsky et al. (1996) however argued that there is a significant difference between understanding parity of integers and understanding quotient groups. In our design work, our goal was to find a way to support students in leveraging their understanding of parity to reinvent the quotient group concept in all (or most of) its complexity. Students are asked to consider the group of symmetries of a square and identify something like parity (evens and odds) in that group. They eventually determine that there are three ways to partition this group into two subsets that interact like evens and odds. For example, one can consider the rotations of a square to be "even" and the reflections to be "odd" and produce a table that mirrors the behavior of the even/odd integers (i.e., Rotation Rotation = Rotation, Reflection Reflection = Rotation, Rotation Reflection = Reflection Rotation = Reflection.) The students are then asked whether these partitions form groups (of two elements that are themselves subsets) and are engaged in attributing meaning to operation given by this 2x2 table. Typically this operation is described as "set multiplication" in the sense that multiplying two of these sets means (left) multiplying every element of the first set by every element in the second set. Next students are asked to form a larger group by partitioning the group of symmetries of a square into smaller subsets. One of the key insights of our early design work has to do with focusing students' activity in a way that supports them in realizing what is needed in order for a partition of a group to form a group of subsets (a quotient group).

Key insight: focusing on the identity subset

Because the set of subsets is supposed to form a group, one of the subsets must act as the identity element of this group of subsets. In our first design experiment focused on quotient groups, we learned that students could figure out what needs to be true of this subset by focusing on how it must behave in order to be the identity element. For example, because the identity element is always idempotent ($ee = e$), the identity subset must be closed under the original group operation. This means that (in the finite case) this subset must be a subgroup. Further, focusing on the identity property ($ge = e$) can support students in inventing a procedure for completing a partition after selecting a subgroup to act as the identity. This procedure can be formalized to define "coset". Then the fact that the identity commutes with every element ($eg = ge$) can support students in developing the idea of normality. This property means that left-multiplying a group element by the subgroup (left coset formation) must yield the same subset as right-multiplying the group element by the subgroup (right coset formation). This process of analyzing partitions that work (and those that do not) with a focus on the identity subset can yield a set of necessary conditions for a partition to form a quotient group (namely that it must consist of cosets of a normal subgroup). These conditions can then be shown to be sufficient, resulting in a formal theory of quotient groups. This entire process is made possible by the fact that the group concept has (at this point in the course) transitioned to a *model-for* more formal activity. Here the group concept is being used to support the new (more formal) activity of developing new kind of group made up of subsets of a group. Simply put, at this point in the course, the students are able to us the group concept as a tool for analyzing the structure of a new object, the set of subsets of a group.

Conclusions

Over the course of several years and a number of design experiments we have attempted to realize Freudenthal's (1973) idea, of taking geometric symmetry as the starting point for an inquiry oriented approach to group theory. Along the way, the design of the course materials and the corresponding instructional theory was profoundly influenced by key insights gained by paying careful attention to the mathematical activity of participating students. In this way, our inquiry into students' activity made it possible to design a course that supports students in reinventing fundamental concepts of group theory by engaging them in authentic mathematical inquiry.

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Modern algebra as an integrating perspective on school mathematics – an interactive genetic and visual approach

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Mathematics teacher education requires concepts of modern mathematics to be understood from an epistemologically rich standpoint. For the case of modern algebra it is demonstrated how in a university course processes of horizontal and vertical mathematization can lead to a deeper understanding of the algrebraic structures underlying core topics of school mathematics.

Looking back at modern algebra

Groups, rings and fields as elements of so called „higher", „modern" or „abstract" algebra are considered a relevant topic for mathematics teacher education (Conference Board of the Mathematical Sciences, 2012). The learning opportunities for modern algebra encountered at university are often „looking ahead", providing tools and concepts for future mathematicians. For future mathematics teachers, however, they should also be „looking back" (Wu, 1997): They should reveal how abstract mathematical concepts originate from concrete problems and how they facilitate a unified view on ideas from school mathematics.

This is the guiding principle for a course in mathematics teacher education (Leuders, 2016), in which students can develop central concepts of modern algebra in a genetic, interactive and visual way, supported by instructional computer software (based on Cinderella and Geogebra, see http://digitales.leuders.net for download). Students find answers to the following questions:

- What are the problems that give rise to central algebraic concepts? (horizontal mathematization)
- Which phenomena/situations/examples (also from school mathematics) can be understood in a universal way? (vertical mathematization)
- Which are the core ideas and the mental models connected to these concepts? (sense-making, mathematical meaning)
- What can be learned about the development of mathematical knowledge? (epistemic reflection)

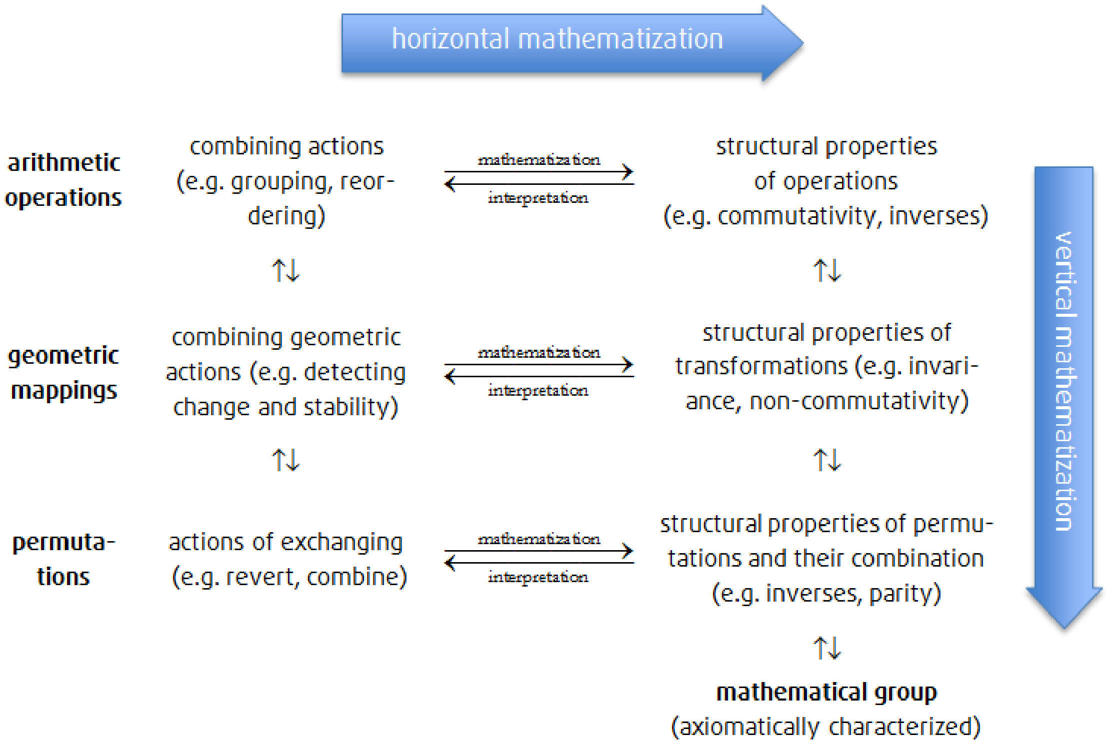
The concept of the course draws on the theory of genetic mathematics learning (Freudenthal, 1991), on a subject matter analyis with respect to mathematical knowledge for teaching (Klein, 1908; Rowland & Ruthven, 2008), on research on learning processes in higher algebra (Asiala et al., 1997; Clark et al., 1997; Larsen et al., 2013) and on principles of visualization of group structures (Carter, 2009). Some of the core aspects are explained in the following.

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings.* Kassel, Germany: Universitätsbibliothek Kassel. urn:nbn:de:hebis:34-2016041950121

Horizontal and vertical mathematization processes

The ability to solve mathematical problems depends on the invidualsʼ use of mental models (van den Heuvel-Panhuizen, 2003). These mental models can be developed during the solution of appropriately chosen ‚geneticʻ problems, a process called *horizontal mathematization* (Treffers, 1978; Freudenthal, 1991, 41). On the other hand one can distinguish processes of *vertical mathematization,* which abstract from the concrete examples, focus on structural aspects, and are characterized by schematization and generalization (ibid.).

Abstract algebra is a well suited example for these categories: It is a conceptual framework that can be constructed in several distinct ways by horizontal mathematization and it requires a crucial step of integrating the concepts by vertical mathematization. For example, problems from arithmetics, geometry or combinatorics all give rise to concepts that can ultimately be integrated into the group concept (Leuders, 2015).



Of course, the categories „horizontal“ and „vertical“ are by no means unequivocally defined, as Freudenthal already conceded (1991, 42): „To be sure, the frontiers of these worlds are rather vaguely marked. The worlds can expand and shrink – also at one anotherʼs expense. Something may belong in one instance to the world of life and in another to the world of symbols [...] For the expert mathematician, mathematical objects can be part of his life in quite a different way but for the novice. The distinction between horizontal and vertical mathematising depends on the specific situation, the person involved and his environment.“

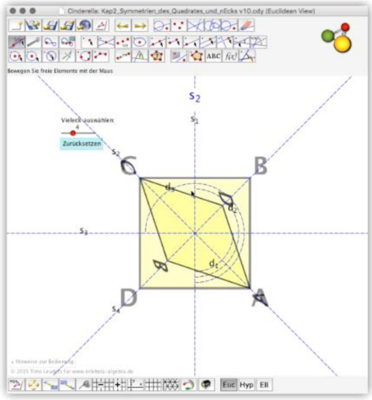
Nevertheless the categories can be helpful as a backbone for teaching modern algebra: They can guide the lecturer to generate a course that displays the inner logic of the development of mathematical concepts and also to make these processes explicit to the students.

This way they are prompted to actively construct mathematics and to reflect on the epistemological processes of the genesis of mathematical concepts.

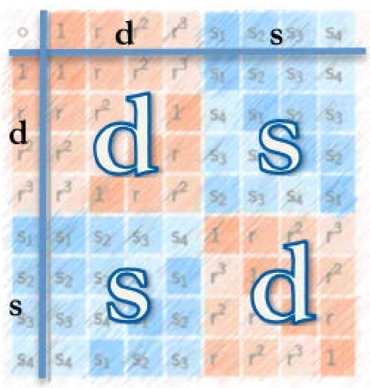
Example problems and student work

To exemplify this, the threefold way towards the group concept, as implemented in the course (Leuders, 2016), is shown below.

(1) The problem of understanding the effect of multiple isometric transformations is explored by using an interactive simulation, resulting in a Cayley table. Some group properties arise in a natural way (inverse and neutral elements, closedness, non-commutativity), some may remain "undiscovered" (assocativity).



∘	1	r	r^2	r^3	s_1	s_2	s_3	s_4
1	1	r	r^2	r^3	s_1	s_2	s_3	s_4
r	r	r^2	r^3	1	s_4	s_1	s_2	s_3
r^2	r^2	r^3	1	r	s_3	s_4	s_1	s_2
r^3	r^3	1	r	r^2	s_2	s_3	s_4	s_1
s_1	s_1	s_2	s_3	s_4	1	r	r^2	r^3
s_2	s_2	s_3	s_4	s_1	r^3	1	r	r^2
s_3	s_3	s_4	s_1	s_2	r^2	r^3	1	r
s_4	s_4	s_1	s_2	s_3	r	r^2	r^3	1



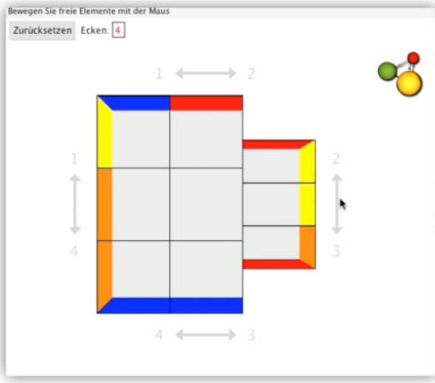
(2) The exploration of the remainder after addition or multiplication of natural numbers yields the structure of arithmetic residue classes, which again show properties already encountered in the previous situation.

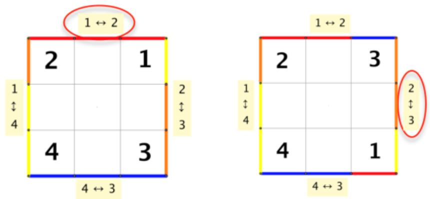
•	0	1	2	3	4	5	6	7	8	9	10	11
0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	0	1	2	3	0	1	2	3
2	0	2	0	2	0	2	0	2	0	2	0	2
3	0	3	2	1	0	3	2	1	0	3	2	1
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	1	2	3	0	1	2	3	0	1	2	3
6	0	2	0	2	0	2	0	2	0	2	0	2
7	0	3	2	1	0	3	2	1	0	3	2	1
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	1	2	3	0	1	2	3	0	1	2	3
10	0	2	0	2	0	2	0	2	0	2	0	2
11	0	3	2	1	0	3	2	1	0	3	2	1

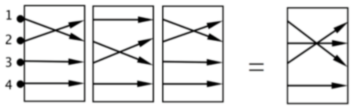
regrouping →

•	0	4	8	1	5	9	2	6	10	3	7	11
0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	2	2	2	3	3	3
5	0	0	0	1	1	1	2	2	2	3	3	3
9	0	0	0	1	1	1	2	2	2	3	3	3
2	0	0	0	2	2	2	0	0	0	2	2	2
6	0	0	0	2	2	2	0	0	0	2	2	2
10	0	0	0	2	2	2	0	0	0	2	2	2
3	0	0	0	3	3	3	2	2	2	1	1	1
7	0	0	0	3	3	3	2	2	2	1	1	1
11	0	0	0	3	3	3	2	2	2	1	1	1

(3) A third exploration is triggered by the question if and how a two dimensional "magic cube" can always be solved. An interactive programms allows for extensive explorations. This leads to permutation structures, which are interpreted also in a geometrical way.

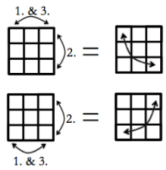






$$(12) \circ (23) \circ (12) = (13)$$

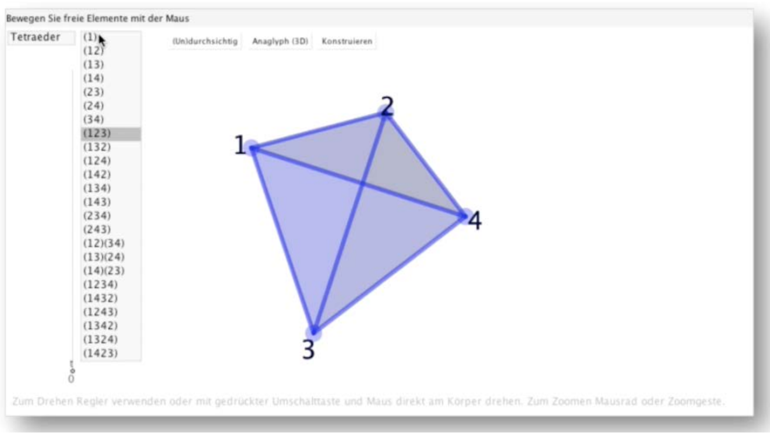
$$(34) \circ (23) \circ (34) = (24)$$



The urge to conceptually integrate the recurring phenonema gives rise to a more formal definition of a group and leads to the question of how to mathematically define the apparent isomorphy of seemingly different structures. In this step horizontal mathematization motivates to (re)create the conventional definitions of groups, subgroups and cosets.

(4) Students are asked to use these new conceptual tools in order to explore groups on an abstract level, using Cayley tables and Cayley diagrams using an interactive computer tool 'group explorer' (top of the following figure).

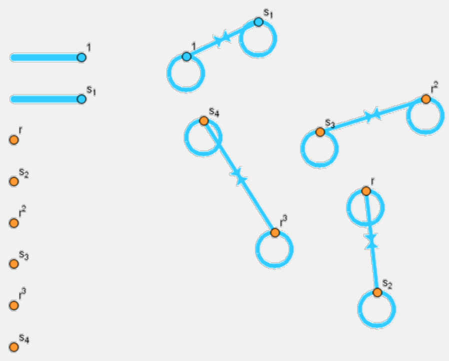
∘	(1)	(1234)	(13)(24)	(1432)	(12)(34)	(24)	(14)(23)	(13)
(1)	(1)	(1234)	(13)(24)	(1432)	(12)(34)	(24)	(14)(23)	(13)
(1234)	(1234)	(13)(24)	(1432)	(1)	(24)	(14)(23)	(13)	(12)(34)
(13)(24)	(13)(24)	(1432)	(1)	(1234)	(14)(23)	(13)	(12)(34)	(24)
(1432)	(1432)	(1)	(1234)	(13)(24)	(13)	(12)(34)	(24)	(14)(23)
(12)(34)	(12)(34)	(13)	(14)(23)	(24)	(1)	(1432)	(13)(24)	(1234)
(24)	(24)	(12)(34)	(13)	(14)(23)	(1234)	(1)	(1432)	(13)(24)
(14)(23)	(14)(23)	(24)	(12)(34)	(13)	(13)(24)	(1234)	(1)	(1432)
(13)	(13)	(14)(23)	(24)	(12)(34)	(1432)	(13)(24)	(1234)	(1)



Cayley diagrams are not very commonly found in agebra courses, but they can help to gain a graphically induced an intuitive insight into group structures. The following excerpt from a student's diary shows the point when she discovers that coset structures may or may not lead to block diagrams – thus opening the door to the definition of normal subgroups.

$(D_4,\circ)$

$\circ$	1	s_1	r	s_2	r^2	s_3	r^3	s_4
1	1	s_1	r	s_2	r^2	s_3	r^3	s_4
s_1	s_1	1	s_4	r^3	s_3	r^2	s_2	r
r	r	s_2	r^2	s_3	r^3	s_4	1	s_1
s_2	s_2	r	s_1	1	s_4	r^3	s_3	r^2
r^2	r^2	s_3	r^3	s_4	1	s_1	r	s_2
s_3	s_3	r^2	s_2	r	s_1	1	s_4	r^3
r^3	r^3	s_4	1	s_1	r	s_2	r^2	s_3
s_4	s_4	r^3	s_3	r^2	s_2	r	s_1	1



Subgroup from <1,s1>:

- ➔ *always two elements are connected*
- ➔ *4 cosets of same size, each consisting of 2 elements*

Question: *Why doesn't a structure like in the group table above exist? [i.e. a structure table in which cosests are arranged in quadratic blocks]*

Other topics such as rings and fields are dealt with in a similar manner, so that at the end of the course even a glance at Galois theory is possible.

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A guided reinvention workshop for the concept of convergence

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This study sketches the design of an additional workshop on a deeper understanding of the concepts of sequences and limits, which some of the students of the class "Analysis 1" attended (experimental group). It also presents the evaluation of the workshop. Therefore a "pre-test" and a "post-test" are used in order to compare the results of the experimental group with them of control groups.

Introduction

German first-year mathematics students usually attend a lecture called "Analysis 1", which differs from calculus lectures. The content is very formal and there is a focus on definitions, theorems and their proofs. This raises the question of whether the misconceptions that are known from secondary students or college students also appear when regarding German university students. Another question is how one can avoid the known misconceptions. To answer these questions I developed a workshop as an optional choice for the students attending the lecture "Analysis 1" for the first time. To evaluate this workshop and to investigate the knowledge of the concept of limits, all students attending the lecture took two tests: one test assessed the previous knowledge relevant for the concept of limit and the second one consists of items regarding sequences and limits. Hence, it is not a real pre-test-post-test design. Therefore, I write "pre-test" and "post-test" with quotation marks.

Some of the studies on students' understanding and their misconceptions of the concept of limit state difficulties with sequences that are not defined by one single formula, errors arising out of the quantifiers, the absolute value and an inequality in the formal definition (Tall & Vinner, 1981; Dubinsky et al., 1988; Cottrill et al., 1996). Many students when first seeing the formal definition intuitively think they have to determine the error ε for a given index N and there is a discrepancy between mathematical language and everyday language (Monaghan, 1991; Roh, 2005). There exist misconceptions that converging sequences have to be monotone, the limit is a lower or upper bound, the limit must not be reached and a sequence can have more than one limit and there is a missing understanding of the difference between the concepts of limit and of cluster point (Davis & Vinner, 1986; Roh, 2005).

Theoretical Background

Tall and Vinner (1981) call the concept definition the definition a person uses for a concept. In contrast, the concept image describes all images, examples, counterexamples etc. a person connects with a concept. Their study shows that learners are highly influenced by the first examples they see of a new concept. These first examples serve as prototypical exam-

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings.* Kassel, Germany: Universitätsbibliothek Kassel.
urn:nbn:de:hebis:34-2016041950121

ples. Therefore they should be carefully chosen in a course so they can cover the whole range of the concept in order to build a wide and proper concept image.

Research Methodology

All students attending the course "Analysis 1" took part in the "pre-test", which took place before sequences and limits were covered in class. Again all students took part in the "post-test" some weeks later after sequences and limits had been treated in detail. The tests were anonymous but by using a code it was possible to match the tests. In between some of the students took part in the workshop in addition to the classes. The rest of the students who attended the class for the first time serve as a control group. Since the workshop was optional and the students had to apply for it, there still can be another possible difference between the two groups: the motivation. Therefore I randomly picked half of the applicants. The ones who were not chosen should serve as a second control group.

The Concept of the Workshop

The workshop consists of two parts: an "introductory part" before sequences and their limits were treated in class and a "follow-up part" afterwards. The aims of the workshop are to build a suitable concept image (Tall & Vinner, 1981) and to prevent the known misconceptions. In the introductory part of the workshop the main aspect is an activity where the students are to reinvent the definition of convergence of a sequence in group work. This exercise is a modification of a teaching sequence for secondary school by Przenioslo (2005). She suggests eleven sequences with limit 1 and one sequence with cluster points 1 and 2, without telling these features to the students. The activity for students is to construct a common property of the eleven sequences that the other sequence does not have. These eleven sequences are well chosen so they include sequences with different characteristics. There are sequences that do not belong to the common (wrong and limited) concept images of convergent sequences but are still convergent in the mathematical sense. The information that the 11 series have a "common property" that 12th series does not have is a strong guidance, a so-called "intentional problem" in the sense of Hußmann (2001) that should enhance the likelihood of correct concept construction. In addition Przenioslo provides some virtual discussions of students regarding the same activity, which can be used to help the students overcome barriers or to focus their attention on certain aspects (Przenioslo, 2005). I adapted this activity to the different situation in the workshop. The task description was changed because most of the participants already knew the terms convergence and limit before from learning calculus at school level, yet they had not seen the formal ε-N-definition. Therefore, if I had used the formulation suggested by Przenioslo, the students could have answered the common property is that the sequences have the limit 1. Hence, I added to the task description that the common property is called "convergence to 1". Furthermore, I reduced the number of the convergent sequences from eleven to six in order to not overburden the participants by considering too many sequences at the same time. Yet the sequences are manifold and the known misconceptions are respected. I did not only translate the virtual discussions but I also revised them. Another difference to the suggestion of Przenioslo is that in her class all students discussed together with the teacher. Since the participants of the workshop are heterogeneous, I decided to let the students discuss in groups of four. This enabled me to assist the different groups with different virtual discussions depending on their intermediate result. At the end of the preliminary part of the workshop the students see the formal ε-N-definition of the limit of a sequence if they have not constructed it themselves.

In the follow-up part of the workshop the participants work with manifold examples and the relations between the concept of limit and other concepts are reviewed. One issue of the workshop is the reflected appliance of several different methods to check if a sequence convergences and if so, to which limit. These methods are not only stated and used but are discussed and additional information such as common difficulties are given. Another activity is the collection of manifold examples and counterexamples of convergent sequences by the participants. An additional issue is working on some simple proofs relating to the concept of limit. The aim of the follow-up part of the workshop is to reflect and strengthen the contents of class learning and to support the wide and proper concept image intended by the introductory part of the workshop.

"Pre-test" and "Post-test"

There are a lot of multiple choice items but in order to get more information and to avoid randomly made decisions explanations of choices were required mostly. The aim of the "pre-test" is to assess previous knowledge relevant for the concept of limit. In my view this contains in particular the concept of function, infinite sets, the handling of terms with fractions and absolute values, inequalities, real numbers and general mathematical argumentation skills. So the "pre-test" contains items to each of these topics. In addition there are items where they have to describe the behavior of a sequence when n becomes large. The "post-test" has to cover all aspects of the concepts of sequences and their limits and it has to assess the aims of the workshop in particular. So items were constructed where the students have to decide if simple sequences in different representations converge and if so, what the limit is. Furthermore there are several items regarding a deep understanding of the concept of limit and in particular the definition and some items assess connections between the concept of limit and other concepts covered in class.

The sample size was 164 for the "pre-test" and 124 for the "post-test". There are a lot of items where the rating is stepped, for example multiple choice items with required explanation and open-ended items. The answers to all open-ended questions are assessed by two raters. The inter-rater reliability given by Cohens' Kappa lies between 0.617 and 1.000 and can be denoted as acceptable to good. The tests are evaluated according to Item Response Theory and are scaled separately. For the "pre-test" and the "post-test" I chose a one-dimensional partial credit model and a two-dimensional partial credit model respectively. The analysis of model fit makes good results in each case.

Results

My aim here is to evaluate the workshop by comparing the experimental with the control groups. Due to breaking up, illness and other reasons there were only ten students who attended both parts of the workshop and also took both tests. In the small control group the loss of members is even higher: only six of the applicants who were not picked wrote both tests. For this reason it is not possible to establish an effect of the workshop when comparing the results of the tests of the small control group and the experimental group. So for exploring an indication of an effect of the attendance at the workshop I compare the participants of the workshop with all other students of the class who took part in both tests. By excluding the students who attend the class for the second time there remains a control group of 77 students. When regarding the whole “post-test”, it is not possible to establish a significant effect. One could reason that the workshop had no success measurable by the test. However, as mentioned before the “post-test” was constructed not only with the aim to evaluate the workshop but it also had to cover all aspects of sequences and their limits, even those that were not discussed in the workshop. Therefore, I constructed a subtest consisting of those items of the “post-test” that directly assess the known misconceptions regarding the concept of limit, which the workshop aimed at. When regarding the subtest, in fact there is a significant effect even though it is much smaller than the influence of the previous knowledge tested by the “pre-test”.

Discussion and Future Research

This study presented a guided reinvention workshop for the concept of convergence. It was possible to show a small but significant effect of the attendance. Why is it so hard to achieve an effect? Disregarding the problem of the shrinking of the groups another problem is the distance of several weeks between the two tests. Apart from the workshop there are many other things which take place in between and influence the students’ performance in the “post-test”.

All discussions recorded during the workshops will be analyzed in order to get more insight in the students’ development of understanding of the concept of limit while exploring the formal definition, their misconceptions and their knowledge after the engagement with the concept. At this analysis the method used by Oehrtman et al. (2011) will be adapted.

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Undergraduates' attempts at reasoning by equivalence in elementary algebra

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Reasoning by equivalence is a formal symbolic procedure where an algebraic expression is manipulated to generate a new and equivalent expression. Reasoning by equivalence is a particularly important algebraic activity in elementary mathematics. This paper reports results from a study in which 147 students attempted to solve two equations in which there are "traps" for the unwary. Few students used any logical connectives, few students checked their answer, and few students showed awareness of domain conventions. Based on an analysis of students' work, I discuss implications of these findings to the design of technology which enable algebraic symbolic manipulation.

Introduction

My experience as a teacher strongly suggests that university students' written work in algebra typically (i) contains no logical connectives or little other justification and (ii) entirely ignores natural domain conventions. When solving an equation students work line-by-line, but each line is apparently disconnected from the previous lines. Despite this experience there was very little research evidence available reporting students' attempts at solving algebraic problems.

Reasoning by equivalence is a formal symbolic procedure where an algebraic expression is manipulated to generate a new and equivalent expression, e.g. a term within an algebraic expression is identified and then replaced by an equivalent term. Reasoning in this way we generate a new problem having the same solutions, and we continue until a "solved" form is reached. A recent survey (Sangwin & Kocher 2016) looked at the extent to which the assessment of current examinations could be automated using the STACK assessment software, see (Sangwin, 2013). One result from this research is that approximately a third of the method marks for current final high school mathematics examinations, such as the International Baccalaureate (IB), are awarded for reasoning by equivalence. There are other forms of reasoning, e.g. using calculus operations to find extreme values, or estimation and implication arising from working with inequalities, but reasoning by equivalence is of central importance. Furthermore, reasoning by equivalence forms the basis of formal proof, e.g. proof by induction and some proofs in real analysis contain reasoning by equivalence.

One underlying motivation for this research is the desire to extend the range of questions which can be automatically assessed in a valid way. Examples of existing software which facilitate reasoning by equivalence are described in (Nicaud, Bouhineau, & Chaachoua, 2004), and (Heeren & Jeuring, 2014). This software doesn't quite manage to capture current practice. For example, in MathXpert (Beeson, 1989) a student indicates what they would like to do and a computer algebra system (CAS) undertakes the calculation for them. A fur-

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ther longer term aim of my research is to design a CAS which supports effective algebraic reasoning. Currently, most CAS promote working line by line in traditional ways. This is problematic because elementary algebra contains a number of subtle “traps”. These traps include division by zero, or gaining/loosing solutions by squaring/square rooting both sides of an equation. Often CAS do not alert a user to the potential for these problems letting the user proceed regardless. Many of these problems are related to the natural domain of definition of the expression being manipulated, e.g. see (Sangwin, 2015) for a selection of examples.

To what extend do students use (and correctly use) logical connectives between lines of algebraic working? What other justification is evident other than “implied equivalence”? To what extent do students check domain constraints? These are the questions the research reporeted here sought to investigate.

Methodology

First year undergraduate students were asked to solve the following two equations, both of which contain algebraic traps.

Question 1: solve $\frac{x+5}{x-7}-5=\frac{4x-40}{13-x}$

Question 2: solve $\sqrt{3x+4}=2+\sqrt{x+2}$

The cohort were a group of 175 students taking an engineering programme at a good United Kingdom university. The methodology relied on students solving these two equations and writing answers on a pro-forma containing only the question at the top of each side of an otherwise blank page. It is standard practice in university mathematics to ask incoming students to sit a short mathematics test as part of normal teaching in the first week at university. The two questions were presented to students as part of such a testing process. Participation was compulsory as part of normal teaching and hence an authentic experience. After the task was completed students were asked if they would volunteer to participate in this study and reasons for participation fully explained. 147 students (84%) agreed to do so and submitted their worksheets for analysis. There was no deception involved, since these tasks are appropriate for these students and full worked solutions were provided to the whole cohort as part of feedback to the whole test.

The first question, taken from (Northrop, 1945, pg. 81), involves the potential for division by an algebraic term which later turns out to be zero. The second task, taken from (Newman & et.al., 1957, pg. 8), involves the potential for spurious roots which arise from squaring both sides of an equation.

Results

All students’ work was assessed by hand and assigned a code which described the form of the main argument. Further codes recorded any use of logic, natural domains of definition and any evidence of checking. Each script was assigned a unique number and the codes, together with the number of lines of working used for each problem, were entered into bespoke software for analysis. Table 1 shows achievement data, both the number of students (and percentage), for question 1 and question 2. The majority of students eventually got correct (C) final answers. Incorrect final answers (I) were more common for question 2. A small number of students left their answers unfinished (U) or omitted question 2 (O). The totals for each question are also given. The mean number (#) of lines of working used #μ together with the standard deviation of the number of lines #σ are also listed.

		Question 2											
		C		I		U		O		Total		#μ	#σ
Question 1	C	76	(51.7%)	22	(15.0%)	9	(6.1%)	6	(4.1%)	113	(76.9%)	10	4.5
	I	8	(5.4%)	9	(6.1%)	3	(2.0%)	3	(2.0%)	23	(15.6%)	10.7	3.6
	U	4	(2.7%)	1	(0.7%)	1	(0.7%)	5	(3.4%)	11	(7.5%)	11	3.7
	Total	88	(59.9%)	32	(21.8%)	13	(8.8%)	14	(9.5%)	147			
	#μ	14.1		7.3		8.2							
	#σ	4.6		3.4		6.4							

Table 1: Achievement data for both question 1 and question 2

For question 1, of the 113 correct responses, 53 (46.9%) cross multiplied, expanded out all brackets and solved the resulting equation correctly to get the unique answer x = 10. None of these students had the opportunity to cancel the term $4x$ - 40 on both sides. Multiplying out in this way before gathering terms on one side of an equation is an entirely safe way to proceed, but in other situations higher powers would make such a procedure infeasible. A further 25 (22%) of students started by writing the left hand side as a rational expression. Of these 22 had the clear opportunity to cancel a factor, but chose not to do so. Instead they cross multiplied and expanded out the brackets. Note that in a previous pilot with more experienced students (and staff), a higher proportion of people seemed to cancel $4x$ - 40 and hence end up with the contradiction 7 = 13. Perhaps this is a mistake only experts make?

For question 1, only 14 (9.5%) of students showed any evidence of logical connectives between algebraic statements. Only 2 students wrote any evidence of having performed a check, and only 1 student explicitly considered domains of definition of the rational expression by excluding x = 7 and x = 13 from the domain of definition for the equation. There is no intersection between the coding, so only 17 (11.6%) of students wrote any evidence of more than algebraic symbolic manipulation.

For question 2, the modal answer for this question consisted of squaring both sides, rearranging and squaring again before solving the resulting quadratic to derive the roots x = 7 or x = -1. Students who did only this were judged as correct (C) even though it is incomplete, and 88 responses took this approach. Only 24 students also checked that their values satisfied the original equation, giving complete and correct solutions by only 16% of the cohort. For question 2, only 4 students showed any evidence of checking domains of definition and only a further 3 students used any logical connectives. The most common mistake was squaring a binomial term incorrectly, e.g. $(\sqrt{a}+\sqrt{b})^2 = a+b$. The following solution occurred 18 times:

$$\begin{aligned} \sqrt{3x+4} &= 2+\sqrt{x+2} \\ 3x+4 &= 4+x+2 \\ x &= 1 \end{aligned}$$

Discussion/Conclusion

Even for comparatively elementary problems, students are taking on average 10 or 14 lines respectively to achieve a correct solution. This number of lines argues for the need for some working to be captured by automatic assessment systems. This strongly suggests that online assessment systems, such as STACK, need to assess more than the final answer, particularly in a formative setting.

Over the last five hundred years there has been a gradual increase in the use of algebraic symbolism. This has enabled very efficient computation, and for complex ideas to be compressed. George Boole, (Boole, 1847) carried this programme forward into reasoning and logic. Note, before these developments calculations and reasoning where rhetorical. Reasoning by equivalence is an important activity which combines both symbolic calculations and logical reasoning, and yet our results provide evidence that students concentrate on the symbolic calculation and almost entirely ignore the reasoning.

One argument in favour of using CAS is that it relieves students from “tedious calculations” and frees up cognitive load for monitoring. However, few students provided much evidence of monitoring. Students may have never been taught to consciously write their reasoning. If so, and if such reasoning is agreed to be important, then teaching practice would need to shift to teach this activity explicitly. Note, however, that about a third of the marks in current IB examinations are already given for reasoning by equivalence and so high-stakes examinations do already reward this activity. Marks for evidence of using a correct method are not necessarily awarded for explicit evidence of reasoning, such as a correct justification for each line, instead examinations condone equivalence reasoning implied by correct adjacent lines of working as the students here have used.

The appropriate use of logical connectives, such as implication and equivalence symbols, link individual algebraic expressions into a single complete entity: a mathematical argument. A single entity could contain domain information and logical connectives. In elementary algebra it is possible to automatically establish the correctness of such arguments. However, computer algebra design mirrors contemporary practice: both enable line by line working with no explicit connection. This style both facilitates errors and makes such errors hard to spot. Both problems might be ameliorated through better CAS design. It will require a close collaboration between school teachers, CAS designers and teachers in higher education, together with their students, to achieve any substantial changes in this situation.

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A comparison of proof comprehension, proof construction, proof validation and proof evaluation

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This paper considers how proof comprehension, proof construction, proof validation, and proof evaluation have been described in the literature. It goes on to discuss relations between and amongst these four concepts—some from the literature, some conjectural. Lastly, it considers some related teaching implications and research.

Introduction

In the mathematics education research literature on proof and proving, there are four related concepts: proof comprehension, proof construction, proof validation, and proof evaluation. There has been little research on how these four concepts are related. We first briefly describe these four concepts, then we consider how they are related. That is, how are they the same? How are they different? Finally, we discuss some related teaching implications and research.

The four concepts as described in the literature

Proof comprehension means understanding a textbook or lecture proof. Mejia-Ramos, Fuller, Weber, Rhoads, and Samkoff (2012) have provided an assessment model for proof comprehension, and thereby described proof comprehension in pragmatic terms. Their model includes both local comprehension and holistic comprehension. *Local comprehension* includes: Writing the theorem statement in your own words. Knowing the definitions of key terms. Knowing the logical status of the statements in the proof. Knowing the kind of proof framework (e.g., direct, contrapositive, contradiction, induction). Knowing how/why each statement follows from previous statements (e.g., making implicit warrants explicit). *Holistic comprehension* includes: Being able to summarize the main, or key, ideas of the proof. Identifying subproofs and how they relate to the overall structure of the proof. Instantiating difficult parts of the proof with an example to aid comprehension. Providing a summary of the proof. Using the ideas from the proof in another proof.

Proof construction (i.e., proving) means attempting to construct correct proofs at the level expected of university mathematics students (depending upon the year of their program of study). What is needed for successful proof construction? To date, more is known in the research literature about difficulties that often prevent students from proving a theorem (e.g., Selden & Selden, 2008; Weber, 2001) than about interventions that would help students' proving.

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Proof validation has been described as the reading of, and reflection on, proof attempts to determine their correctness. Some validation studies have been conducted with undergraduates and mathematicians (e.g., Inglis & Alcock, 2012; Selden & Selden, 2003; Weber, 2008). The broad general finding is that undergraduates check "surface features" of proofs such as equations, whereas mathematicians look for the logical structure and the correctness of implied warrants.

Proof evaluation has been described by Pfeiffer (2011) as determining whether a proof is correct and "also how good it is regarding a wider range of features such as clarity, context, sufficiency without excess, insight, convincingness or enhancement of understanding." (p. 5). However, in order to distinguish proof evaluation from proof validation, we will put aside the portion referring to validation and concentrate on features of proofs including clarity, context, convincingness, beauty, elegance, and depth (e.g., Inglis & Aberdein, 2015). We would also like to separate proof evaluation from the use of adjectives that we have found with student validations, where terms like "wacky" and "confusing" were used when evaluating other students' proof attempts (Selden & Selden, 2015).

The paucity of research on the interrelationships

To date, there does not seem to have been much research attempting to relate the four concepts. Here is what we have found: Pfeiffer (2011) conjectured that practice in proof evaluation, as she defined it, could help undergraduates appreciate the role of proofs and also help them in constructing proofs for themselves. She obtained some positive evidence, but her conjecture needs further investigation. Selden and Selden (2015) obtained some evidence that improving undergraduates' proof construction abilities would *not* necessarily enhance their proof validation abilities and suggested that proof validation needs to be explicitly taught.

Relationships between and amongst these four concepts

Proof comprehension

Mejia-Ramos, et al. (2012), in their assessment model, considered both local comprehension/understanding and holistic understanding of a proof. By *local comprehension*, they meant knowing the definitions of key terms, knowing the logical status of the statements in the proof, knowing the proof framework, and knowing how/why each statement followed from previous statements. Such local comprehension is also needed for proof validation as described by Selden and Selden (2003); see below.

By *holistic comprehension*, Mejia-Ramos, et al. (2012) meant being able to summarize the main ideas of the proof, identifying the modules [subproofs] and how they relate to the proof's structure, being able to transfer the ideas of the proof to other proving tasks, and instantiating the proof with examples. Being able to summarize the main ideas of a proof and identifying modules [subproofs] are also useful for proof validation, but instantiating parts of a proof with examples to check a result is rarely done by students. However, in this regard, Weber (2008) found that some mathematicians did so when checking congruences in number theory proofs. Also, being able to transfer the ideas of a proof to other proving tasks has more to do with generalization of a proof's techniques—something not needed for proof validation.

Weber (2015) found five strategies that good 4th year university mathematics students use to foster proof comprehension. These are "(i) trying to prove a theorem before reading its proof, (ii) identifying the proof framework being used in the proof, (iii) breaking the proof into parts or subproofs, (iv) illustrating difficult assertions in the proof with an example, and (v) comparing the method used in the proof with one's own approach" (p. 289) and suggested there might be more. Also, in a larger, internet follow-up study reported in the same paper, it was found that most mathematicians wanted their students to implement these five strategies.

Can students be taught these strategies? Samkoff and Weber (2015) attempted to teach these five strategies, using reciprocal teaching, and found a qualified "yes". Instantiating a theorem statement with an example helped students understand its proof. Students were also able to identify proof methods, especially if they looked at the proof's assumptions and conclusions. However, students did not instantiate a line of a proof with a specific example. In addition, Samkoff and Weber found that simply asking students to "know the definitions of the terms in the theorem" was not enough. Moreover, simply asking students how to prove a theorm before reading its proof lead to superficial responses (e.g., "use epsilons").

Furthermore, it seems that how one reads a proof depends on what one wants to "get out of it" (Rav, 1999). Indeed, Mejia-Ramos and Weber (2014) found that mathematicians commonly read published proofs to gain insight, not to check their correctness, and additionally, that mathematicians consider refereeing a proof to be a substantially different activity.

Proof construction

We limit our consideration to situations in which undergraduates are asked to prove theorems, not to conjecture them, as this is the more common situation in U.S. undergraduate mathematics education. What is needed for successful proof construction? It is not clear that this has been discussed much in the mathematics education research literature. However, the kinds of difficulties that can stop students from proving a theorem have been researched. These include: Difficulties interpreting and using mathematical definitions and theorems. Difficulties interpreting the logical structure of a theorem statement one wishes to prove. Difficulties using existential and universal quantifiers. Difficulties handling symbolic notation. Knowing, but not bringing, appropriate information to mind. Knowing which (previous) theorems are important (e.g., Selden & Selden, 2008: Weber, 2001).

One overlap of proof construction with both proof comprehension and proof validation seems to be in knowing and using definitions and theorems appropriately. For proof construction, one needs to bring definitions and theorems to mind at an appropriate time so one can use them. However, in proof comprehension and proof validation, definitions and theorems have already been invoked, so one does not have to think of them, rather one only has to decide if they have been used appropriately. In general, it would seem that creating a new proof oneself, would be harder than merely comprehending what has already been done by someone else or checking its correctness, provided it is not a "garbled" student proof attempt.

Proof validation

While proof validation has been described briefly as the reading of, and reflection on, a proof attempt to determine its correctness, much is involved. Selden and Selden (2003) elaborated on what it might take to validate a proof attempt, suggesting that doing so is more complex than simply reading from the top-down:

- Validation can include asking and answering questions, assenting to claims, constructing subproofs, remembering or finding and interpreting other theorems and definitions, complying with instructions (e.g., to consider or name something), and conscious (but probably nonverbal) feelings of rightness or wrongness.
- Proof validation can also include the production of a new text—a validator-constructed modification of the written argument—that might include additional calculations, expansions of definitions, or constructions of subproofs.
- Towards the end of a validation, in an effort to capture the essence of the argument in a single train-of-thought, contractions of the argument might be undertaken (p. 5).

If one compares this statement on proof validation with the Mejia-Ramos, et al. (2012) assessment model for proof comprehension, there seem to be several possible common features: Knowing the definitions of key terms. Checking the logical status of statements. Knowing which proof framework was used. Constructing subproofs. Perhaps summarizing the proof. But, the relation to considering examples is not so clear. However, in this regard, Weber (2008) found that his eight mathematicians used example-based reasoning in proof validation, that is, they often checked the truth of an implied warrant through use of a carefully chosen example. It may be that many mathematicians, through experience, have developed implicit knowledge of which examples are likely to be useful.

One big *difference* between proof comprehension and proof validation might be that in most proof comprehension situations one can reasonably assume a proof is correct, especially if it appears in a lecture or textbook. Indeed, one's skepticism about the validity of a proof may depend greatly upon its source—whether from a textbook, a journal, a colleague, or a student. On this issue, Samkoff and Weber (2015) concluded, "It would not be surprising if strategies for [proof] validation differed from those of [proof] comprehension."

Proof evaluation

As described above, proof evaluation seems more like making value judgments about a finished proof or a published proof text. When a student's proof attempt is being examined by another student, such judgments can be about not understanding what is written, rather than about its beauty, clarity, elegance, or depth. In the recent Selden and Selden (2015) validation study, students said they found parts of the proof attempts "confusing", "convoluted", or "a mess". One student found the notation "wacky". Other student validators said too much or too little information was given in a proof. Thus, for students, it seems that "making sense" of (i.e., understanding/comprehending) a proof attempt (as written) is a prerequisite for proof validation to begin.

In an internet study, Inglis and Aberdein (2014) asked 255 mathematicians to consider whether a proof of their own choosing was “elegant”, “insightful”, “explanatory”, “polished”, and so forth. The mathematicians were provided 80 such adjectives. The authors concluded that mathematicians’ adjective choices could be classified along four dimensions: aesthetics, intricacy, utility, and precision. Additionally, we conjecture that evaluations such as those made by these mathematicians would require a certain familiarity with, and competence with, proof comprehension and proof construction. We feel one would need to have seen (i.e., comprehended) and constructed many proofs in order to make value judgments on characteristics such as elegance, insightfulness, and depth.

While naïve student judgments about whether a is proof “confusing” are often personal and idiosyncratic, these might sometimes also be a characteristic of how a proof was written. Proofs are written in a certain genre (Selden & Selden, 2013) and advice is often given to both student and mathematician authors on how to write them (e.g., Tomforde, n.d.). In our “proofs” course (Selden, McKee, & Selden, 2010), we first validate students’ proof attempts, then go over them again to comment on their style (i.e., their adherence to the genre of proof).

In sum

There are more questions here than answers. One can not only ask, how are proof comprehension, proof construction, proof validation, and proof evaluation related, but also how does one teach them? Which should be taught first or should they be taught in some combination? What is the effect of doing so?

It would seem that students’ proof comprehension would benefit from their attempts at proof construction and vice versa--suggesting these two concepts/skills should be taught together. Indeed, reading comprehension researchers (e.g., McGee & Richgels, 1990) state that reading and writing taught together result in better learning. In addition, before submitting a proof, whether for homework or a journal, one needs to validate it for oneself to ensure its correctness. Finally, it would seem that one should have a good grasp of the first three--proof comprehension, proof construction, and proof validation--before attempting to evaluate proofs as beautiful, elegant, insightful, obscure, and so forth.

Related teaching implications and research

What do mathematicians consider when preparing pedagogical proofs? What do students “get out of” proofs presented in lectures or textbooks? How can one teach proof comprehension?

There has been some research on each of the above. While clearly informative, this research has not specifically considered the relationship of proof comprehension to proof construction, proof validation, or proof evaluation. For example, Lai and Weber (2014) found that mathematicians said that they considered both the intended audience and medium, whether lecture or textbook, in their proof presentations. However, they also found that although mathematicians valued pedagogical proofs featuring diagrams and emphasizing main ideas, they did not always incorporate these into the pedagogical proofs they constructed or revised.

Researchers are interested in proof comprehension because mathematics undergraduates, at least at the upper-division level in the U.S., spend a lot of time watching and listening to proofs being demonstrated in lectures and are also assigned proofs to read in their textbooks. The question is: What do, and what should, students "got out of this"? To begin to answer this question, Fukawa-Connelly, Lew, Mejia-Ramos, and Weber (2014) examined what students "got out of" one real analysis professor's proof of the theorem that if a sequence has the property that the distance between any two consecutive terms x_n and x_{n-1} is less than r^n, where $0 < r < 1$, then it converges. The professor's lecture was much more detailed than what he wrote on the blackboard, but most students only copied down what was on the blackboard, and did not pay attention to the professor's added oral remarks. As a result, the students did not comprehend much of what the professor intended to convey. Apparently, the students, unlike the professor, did not see the professor's oral explanations as important.

In order to investigate the feasibility of teaching proof comprehension using self-explanation training, Hoods, Alcock and Inglis (2014) conducted three experiments. Their self-explanation training was designed to focus students' attention on logical relationships within mathematical proofs. The first two experiments were small scale. Students who had the self-explanation training tended to generate higher quality explanations and performed better on a comprehension test constructed according to the assessment principles of Mejia-Ramos, et al. (2012). The students also increased their cognitive engagement. Experiment 3, with 107 students in a lecture situation, showed that 15 minutes of reading a self-study intervention booklet, describing self-explanation, also improved students' proof comprehension, and this improvement persisted over time, suggesting proof comprehension can be taught effectively.

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Proof construction perspectives: structure, sequences of actions, and local memory

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This theoretical paper considers several perspectives for understanding and teaching university students' autonomous proof construction. We describe the logical structure of statements, the formal-rhetorical part of a proof text, and proof frameworks. We view proof construction as a sequence of actions, and consider actions in the proving process, both situation-action pairs and behavioral schemas. We call on several ideas from the psychological literature and introduce the concept of local memory – a subset of memory that is partly activated during prolonged consideration of a proof.

Introduction

Question

What should be taught to university students who want to learn proof construction? One answer is: The content of some subfields of mathematics, such as linear algebra or real analysis. That is, theorems, explanations of their proofs, plus some intuition about those subfields, with student proving relegated mainly to homework and tests. We suspect this answer is close to how many mathematicians themselves were taught, and this itself is evidence that such teaching is sometimes, perhaps often, effective. There is another reason the above answer might be favored. We recall proving a nice result in field A using a result from an "unrelated" seminar in field B. This kind of serendipitous proof experience probably happens often enough to suggest it is valuable for students to take courses covering a wide variety of mathematical content. However, students just learning to construct proofs are not in a position to use such serendipitous experiences. For many students the teaching of mathematical content contains too little proving practice to be adequate for developing beginning proof construction skills.

A second answer to the above question comes from observing students' proof construction attempts and noting what prevents them from succeeding. Of course, mistakes can, but what else about the proof construction process, other than the content of mathematics, can prevent success? What follows describes perspectives that elaborate this second answer. These perspectives might contribute to a "content of proof construction" that will provide an alternative approach to teaching and learning proving. These perspectives will often be abstracted above the level of mathematical content--for example, difficulty with definitions, as opposed to difficulty with the meaning of normal subgroup. Before describing the perspectives, we describe the course from which some of the ideas emerged.

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The course

The theoretical perspectives described below emerged from the past ten years of teaching/designing a course for beginning mathematics graduate students who felt they needed help with proving. We saw, and still see, autonomous proof construction as an activity, like learning a sport, that is mastered largely through doing it, perhaps with some coaching. Thus, in our "proofs course" we maximized student proof construction experiences. We and several mathematics education graduate students collected field notes and videos of these classes and analyzed them. We were looking for ways to help students learn to autonomously construct proofs, and the mathematical content involved was only a means to that end. In order to include a variety of kinds of proofs that students might write in subsequent courses, we included sets, functions, a little real analysis, some abstract algebra (semigroups), and if there was time, some topology. There were no lectures and the 4-10 students, presented their proof attempts at the board in class. For each proof attempt, we provided a, sometimes extensive, critique. Occasionally there were explanatory side comments, such as on logic. For more information, see Selden, McKee and Selden (2010, p. 207).

The proof text

The genre

There are distinctive features that commonly occur in proofs and reduce unnecessary distractions in validation (reading/reflecting on proofs to judge their correctness). These features increase the probability that any errors will be found, thereby improving the reliability of the corresponding theorems. Proofs are not reports of the proving process, contain little redundancy, and contain minimal explanations of inferences. They contain only very short overviews or advance organizers and do not quote entire statements of previous theorems or definitions that are available outside of the proof. Symbols are generally introduced in one-to-one correspondence with mathematical objects. For example, one does not say, "Let $x \in R$. Now let $y = x$." Finally, proofs are "logically concrete" in the sense that, where possible, they avoid quantifiers, especially universal quantifiers. Their validity is often seen to be independent of time, place, and author. Details can be found in Selden and Selden (2013).

The logical structure of statements

Statements, such as theorems or definitions, have a logical structure that can be described as formal or informal. A statement is *formal* if the variables are named; quantifiers are expressed explicitly and typically written first; and logical operators are just the most commonly used ones: *and*, *or*, *not*, *if-then*, and *if-and-only-if*. In addition, a formal statement should not be logically reducible to a shorter one. Otherwise, a statement is informal. Examples are: "Differentiable functions are continuous", and in a semigroup context, "A group has no proper left ideals". A formal version of the later in a semigroup context is: "For all semigroups S and for all left ideals L of S, if S is a group, then L=S", which removes the hidden double negative, "no proper". For more information, see Selden and Selden (in press).

Informal statements are often used to state theorems, perhaps because they are memorable, often short, and psychologically combine easily with other information. However, we have found that beginning university students of proof construction are not likely to be able to reliably unpack them into formal statements (Selden & Selden, 1995). Such unpacking is important for both proof construction and validation (Selden & Selden, 2003). Thus, to build student self-efficacy (Bandura, 1995), it is better at the beginning of a proof construction course to state theorems and definitions formally.

A structure of proof texts

A completed proof text can be divided into a formal-rhetorical part and, its complement, a problem-centered part. The *formal-rhetorical* part is the part that depends only on the logical structure of the statement of the theorem, earlier results, and associated definitions. It does not depend greatly on intuition about, or a deeper understanding of, the concepts involved or genuine problem solving in the sense of Schoenfeld (1985, p. 74). The *problem-centered* part does depend on problem solving, intuition, heuristics, and a deeper conceptual understanding of the concepts involved (Selden & Selden, 2011). We suggest that beginning university students of proof construction are likely to benefit most from constructing proofs that have large formal-rhetorical parts and more advanced university mathematics students are likely to benefit most from those that have large problem-centered parts.

Proof frameworks

A major structure that can contribute to construction of the formal-rhetorical part of a proof text is a proof framework (Selden & Selden, 1995) of which there are several kinds. A *proof framework* is roughly the logical parts of the theorem statement placed in the approximate position they would occur in the completed proof text. Here is an example. Suppose the statement of a theorem has the form "For all $x \in X$, if $P(x)$ then $Q(x)$." Then a proof framework would start: "Let $x \in X$. Suppose $P(x)$. … Then $Q(x)$." where the ellipsis represents a blank space to be filled. In many cases, a (second-level) framework can be constructed for the proof of $Q(x)$ and placed in the blank space of the first framework. In this way, a proof framework is constructed from the top and bottom of a proof towards the middle. The "Let $x \in X$" above means x will be treated as a fixed, but arbitrary constant, rather than a variable, so that the proof construction will depend only on propositional calculus, rather than the harder predicate calculus. For some time students may not *feel* that doing this is appropriate. (See the case of Mary, described in Selden, McKee, and Selden, 2010, p. 209).

Operable interpretations

In writing the formal-rhetorical part of a proof, it can be helpful to associate definitions and previously proved results with *operable interpretations*. These interpretations are similar to Bills and Tall's (1998) idea of operable definitions. For example, given a function $f: X \to Y$ and $A \subseteq Y$, we define $f^{-1}(A) = \{x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa." One might think that this sort of association of a definition with an operable form would be unnecessary. However, we have found that for some students doing so is not easy. Indeed, we have also noted instances in which students have had both a definition and an operable interpretation available, but still did not act appropriately. Apparently, *actually* implementing an operable interpretation is distinct from *knowing* that one *could* implement it.

We suggest that students, or small groups of students, can and should develop some operable interpretations independently of a teacher. However, if or when this should be done in a particular course is a design problem.

Psychological considerations

Much of proof construction and its teaching and learning can be explained, or even guided, by psychological considerations. Here are a few ideas/structures we call on. *Working memory* includes the central executive, the phonological loop, the visuospatial sketchpad, and an episodic buffer (Baddeley, 2000) and makes cognition possible. It is involved in learning and attention and has limited capacity which, when exceeded, produces errors and oversights. There are several kinds of *consciousness* but we will always mean phenomenal consciousness -- approximately, awareness of experience. There are (at least) two systems of cognition that operate in parallel. *S1 cognition* is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory. In contrast, *S2 cognition* is slow, and conscious. It requires attention, is effortful, evolutionarily recent, and burdens working memory (Stanovich & West, 2000). System 2 may monitor System 1 and may sometimes take over. The idea includes that S1 and S2 have some underlying causal structure/mechanism. Furthermore, S1 is probably a system of systems (Stanovich, 2009).

Proof construction as a sequence of actions

Proof construction can be seen as a sequence of actions which can be physical (e.g., writing a line of the proof) or mental (e.g., changing one's focus or trying to recall a relevant theorem). A sequence of all of the actions that eventually leads to a proof is usually considerably longer than the final proof text itself and often proceeds in a different order. For an example of how circuitous this can be, see Dr. G s' ultimately successful proving episode in Selden and Selden (2014). This fine-grained action approach facilitates noticing which beneficial student proving actions to encourage, and which detrimental student proving actions, to discourage.

Each action in a proof construction arises from a *situation* in the partly completed proof. The situation may be interpreted by the prover by drawing on information from long-term memory and a warrant for the acton may be developed. Interpreted situations are mental states and so are unobservable. However, a teacher can often infer an interpreted situation from observing the partly completed proof.

Behavioral schemas

If, during several proof constructions in the past, similar situations have corresponded to similar reasoning/warrants leading to similar actions, then, just as in classical associative learning (Machamer, 2009), a link may be learned between them, so that another similar situation evokes the corresponding action in future proof constructions without the need for the earlier warrant or intermediate reasoning. Using such *situation-action links* strengthens them, and after sufficient practice/experience, they can become overlearned, and thus automated (Morsella, 2009). We call automated situation-action links *behavioral schemas* (Selden, McKee, & Selden, 2010).

A person executing an automated action, such as a behavioral schema, tends to: (1) be unaware of any needed mental process; (2) be unaware of intentionally initiating the action; (3) executes the action while putting little load on working memory; and (4) finds it difficult to stop or alter the action (Bargh, 1994). We see behavioral schemas as part of S1, rather than S2.

We also view behavioral schemas as belonging to a person's knowledge base. They can be considered as partly conceptual knowledge (recognizing and interpreting the situation) and partly procedural knowledge (the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment". In using a situation-action link or a behavioral schema, both the situation and the action (or its result) seem always to be at least partly conscious.

Here is a hypothetical example of one such possible behavioral schema that could conserve resources. One might be starting to prove a statement having a conclusion of the form p or q. This would be the situation at the beginning of the proof construction. If one had encountered this situation a number of times before, one might readily take an appropriate action, namely, in the written proof assume not p and prove q or vice versa. While this action can be warranted by logic (if not p then q, is equivalent to, p or q), there would no longer be a need to bring the warrant to mind.

It is our contention that, by forming behavioral schemas, large parts of proof construction skill can be automated, that is, that one can facilitate university students in turning what has been regarded as parts of S2 cognition into S1 cognition. Doing this would make more resources, such as working memory, available for such high cognitive demand tasks as the truly hard problems that need to be solved to complete many proofs.

The idea that much of the deductive reasoning that occurs during proof construction could become automated may be counterintuitive because many psychologists (e.g., Schechter, 2012), and (given the terminology) probably many mathematicians, assume that deductive reasoning is largely S2. We think that for successful students this change now sometimes happens naturally and implicitly, but with teaching, could be greatly enhanced.

It appears that consciousness plays an essential role in triggering the enactment of behavioral schemas for constructing proofs. This is reminiscent of the role consciousness plays in reflection. It is hard to see how reflection, treated as selectively and approximately re-presenting past experiences in a new order, could be possible without first having had the experiences. We have developed a six-point theoretical sketch of the genesis and enactment of behavioral schemas (Selden, McKee, & Selden, 2010, pp. 205-206). (1) Behavioral schemas are immediately available. They do not normally have to be remembered, that is, searched for and brought to mind before their application. This distinguishes them from most conceptual knowledge and episodic and declarative memory, which generally do have to be recalled or brought to mind before their application. (2) Behavioral schemas operate outside of consciousness. One is not aware of doing anything immediately prior to the resulting action – one just does it. Thus, a behavioral schema that leads to an error is not under conscious control and merely being shown a counterexample might not prevent future reoccurrences. (3) Behavioral schemas tend to produce immediate action. One becomes conscious of the action resulting from a behavioral schema as it occurs or immediately after it occurs. (4) One might reasonably ask, can several behavioral schemas be "chained together" outside of consciousness, as if they were one schema? For most persons, this seems not to be possible. If it were so, one would expect that a person familiar with solving linear equations could start with $3x + 5 = 14$, and *without bringing anything else to mind*, immediately say $x = 3$. We suggest that very few (or no) people can do this. (5) An action due to a behavioral schema depends on at leat some conscious input. In general, a stimulus need not become conscious to *influence* a person's actions, but such influence is normally not precise enough for doing mathematics. (6) Behavioral schemas are acquired (learned) through (possibly tacit) practice. That is, to acquire a beneficial schema a person should actually carry out the appropriate action correctly a number of times – not just understand its appropriateness. Changing a detrimental behavioral schema requires similar, perhaps longer, practice.

Feelings and proof construction

The words "feelings" and "emotions" are often used more or less interchangeably. Both appear to be conscious reports of unconscious mental states, and each can, but need not, engender the other. We will follow Damasio (2003) in separating feelings from emotions with emotions expressed by physical states, such as temperature, facial expression, blood pressure, pulse rate, perspiration, and so forth, while feelings are not (Damasio, 2003, pp. 67-70). That is, feelins are conscious mental states, rather than physical states. Feelings such as a feeling of knowing can play a considerable role in proof construction (Selden, McKee, & Selden, 2010). For example, one might experience a feeling of knowing that one has seen a theorem useful for constructing a proof, but not be able to bring it to mind at the moment. Such feelings of knowing can guide cognitive actions; for example, they can influence whether one continues a search or aborts it (Clore, 1992, p. 151). We call such feelings that can influence cognition *cognitive feelings*. When we speak of feelings here, we mean non-emotional cognitive feelings.

For the nature of feelings, we follow Mangan (2001), who has drawn somewhat on William James (1890). Feelings seem to be summative in nature and to pervade one's whole field of consciousness at any particular moment. For example, to illustrate what it might mean for a feeling to pervade one's whole field of consciousness, consider a hypothetical student taking a test with several other students in a room with a window. If, at a particular time, the student looks at his test, then towards the other students, and finally out of the window, at each of the three moments he or she perceives external information from only that moment. But if the student feels confident (i.e., has a feeling of knowing) that he or she will do well on the test during one of these moments, then he or she will also feel confident during the other two. This suggests that feelings are widely available to be focused on and can directly influence action.

Additional nonemotional cognitive feelings, different from a feeling of knowing, are a feeling of familiarity and a feeling of rightness. Mangan (2001) has distinguished these. Of the former, he wrote that the "intensity with which we feel familiarity indicates how often a content now in consciousness has been encountered before", and this feeling is different from a feeling of rightness. It is rightness, not familiarity, that is "the feeling-of-knowing in implicit cognition". Rightness is "the core feeling of positive evaluation, of coherence, of meaningfulness, of knowledge". In regard to a feeling of rightness, Mangan has written that "people are often unable to identify the precise phenomenological basis for their judgments, even though they can make these judgments with consistency and, often, with conviction. To explain this capacity, people talk about 'gut feelings', 'just knowing', hunches, [and] intuitions". Often such quick judgments (i.e., the results of S1 cognition) can be correct, but they sometimes need to be checked, that is, S2 cognition needs to "kick in" and override such incorrect quick judgments.

Finally, we conjecture that feelings may eventually be found to play a larger role in proof construction than they as yet have. They provide a direct link between the conscious mind and the structures and possible actions of the unconscious mind, which has not been well studied in the proving context.

Local memory

One might think that proof construction consists mainly of communication with others or oneself using speech, vision, etc., or their inner versions (Sfard, 2010). That is, it is mainly conscious. We take a somewhat different view. There appears to be a very large amount of memory maintained outside of consciousness. Conscious information can sometimes influence the activation of related information in memory (i.e., bring something to mind) and sometimes cannot do so (Selden, Selden, Mason, & Hauk, 2000). In constructing a proof, often much more relevant information can be activated than can be simultaneously held in mind. When information that cannot be held in mind is lost from consciousness, it seems not to be returned to its original state, but to a state of partial activation, and hence can be easily recalled. Often, in attempting a long proof, a considerable amount of information is generated and partially activated. We call such partially activated information *local memory*. We have found that we can easily recover such local memory even a day or so after putting aside a proof construction session, provided we have not engaged in some other cognition similar to proving. That is, we can easily "pick up where we have left off".

Memory activation may itself provide a useful flexibility. When information is activated and then returned to memory, it may be slightly altered. After the same information is activated several times, somewhat different information may, seemingly serendipitously, be activated in the next iteration, and consequently, produce new proof ideas.

We hope others will extend and improve these ideas, especially those that call on psychology, which we think has developed in ways useful in mathematics education since the cognitive revolution around the mid 20th century.

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A coherent approach to the fundamental theorem of calculus using differentials

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We describe an approach to introductory Calculus that supports students in connecting their conceptions of derivatives and integrals by incorporating the FTC as a central idea from the first day of the course. To accomplish this goal we re-conceptualize the idea of differential, introducing it before the notion of derivative in the context of constant rate of change in linear variation. In doing so, we view changes in variables happening continuously, as opposed to happening in increments.

Several authors have built introductory calculus courses based on the concept of infinitesimal as introduced in Robinson's (1966) nonstandard analysis. Three prominent examples are Henle (1979), Rogers (2005), and Keisler (2012). They argued, and we agree to a certain extent, that an approach to calculus based on infinitesimals is more intuitive for students than is the more common approach that is based on limits.

Another point of entry into the calculus is through the use differentials *in place of* derivatives (e.g., Dray & Manogue, 2010; Rogers, 2005). Rogers' meaning of a differential seems, to us, to be very much like Robinson's infinitesimal. Dray and Manogue's use of differentials seems to be driven by notational simplicity that they provide. We cannot tell with certainty what Dray and Manogue mean by a differential, but it seems they meant *differential* to be a small change in a quantity. Regarding common meanings of differential in calculus textbooks, we surveyed 17 classic and contemporary calculus textbooks; most of them do not mention differentials at all for single variable calculus, and the few that do, define *differential* after having fully developed the derivative, and they define the differential dy as $dy = f'(x)dx$.

Existing approaches to calculus based on the ideas of infinitesimals, limits, or differentials fail to address an important common shortcoming in calculus students' thinking: students tend to think of variables statically. To them, variables do not vary. Calculus, to students who conceive variables statically, is divorced from ideas of variation, covariation, accumulation, and rate of change—the very ideas that the inventors of the calculus intended to address. White and Mitchelmore (1996), Jacobs (2002), Carlson et al. (2002), and Trigueros and Jacobs (2008) demonstrated the insidious effects on students' understandings of and ability to model dynamic situations and pointed to students' static conceptions of variables as being at the root of their difficulties.

Our final concern with approaches that support students' tendencies to think about variables statically is that the Fundamental Theorem of Calculus (FTC) is fundamental neither to students' understandings of derivatives nor to their understandings of integrals. Instead,

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derivatives are about slopes of tangents, integrals are about areas bounded by a curve, and the FTC, coming after both derivatives and definite integrals, is about neither slopes nor areas. There is nothing fundamental about the FTC in students' thinking.

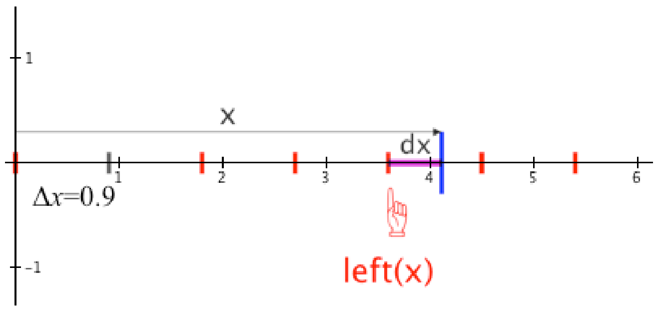
Here we outline our approach to developing the calculus so that it (i) explicitly addresses students' problematic, static meaning of variables, and (ii) supports students in connecting their conceptions of derivatives and integrals by incorporating the FTC as a central idea from the first day of the course. To accomplish this goal we needed to re-conceptualize the idea of differential.[1]

The fundamental theorem of calculus frames our entire course. We explain to students at the outset that the entirety of calculus addresses two foundational problems, namely:

1. You know how fast a quantity is changing at every moment; you want to know how much of it there is at every moment.
2. You know how much of a quantity there is at every moment; you want to know how fast it is changing at every moment.

We found that US college students and Israeli high school students are not prepared to think about these foundational questions profitably. Their image of function is typically a one-number-in one-number-out function machine, and they cannot use function notation representationally. Also, in line with earlier research, students think of variables statically. To them, a variable's value varies by substituting different numbers in its place—one number at a time. Accordingly, their understanding of the continuum (the real number line) is that it is composed of integers, a smattering of rational numbers, and 7-10 irrational numbers. Finally, their understandings of quantity are limited largely to lengths, areas, and volumes, where areas and volumes are conceptually one-dimensional (Thompson, 2000). As such, continuous variation is not part of their image of a real-valued variable and it requires a concerted effort on students' part to construct continuous variation as a way of thinking.

We address students' ill-preparedness in many ways, focusing on their conceptions of the continuum and on envisioning variables as varying continuously. The image of continuous variation also is an important part of our materials on the concept of function. We also develop the idea of constant rate of change in the guise of linear variation. It is in the context of linear variation that we introduce the idea of differential. When two quantities x and y change at a constant rate with respect to each other, then changes in y vary in proportion to changes in x. Or, $dy = mdx$. That is, we view changes in variables happening continuously, as opposed to changes in variables happening in increments. To this end, we talk about Δx as the length of intervals that partition the x-axis, but we speak of the value of x varying



[1] Background for this approach may be found in (Kouropatov & Dreyfus, 2013, 2014; Thompson, 1994; Thompson, Byerley, & Hatfield, 2013; Thompson & Silverman, 2008).

continuously through any Δx-interval. The value of dx is the difference between the "current" value of x and the beginning (denoted left(x)) of the Δx-interval that contains the current value of x. That is, a differential in x is a variable whose value varies through the interval $(0, \Delta x]$, repeatedly. Therefore, dy is a variable whose value varies through the interval $(0, mdx]$, where m is the constant rate of change that relates changes in y with changes in x within the Δx-interval that contains the current value of x as it varies.

We hasten to point out that we introduce the idea of differential as soon we introduce linear variation. We *do not* base the idea of differential on the idea of derivative.

We then define the concept of a *moment* of a variable as a small interval containing a value of the variable. The idea of a *moment* is best illustrated by the case where the variable is time: taking a photo with the shutter being open for a small interval of time – a moment. Anything moving within the camera's range of view will create a small blur, and this will be true no matter the shutter's setting. The generalization to variables other than time is that *all variation is blurry.* Thus, a moment in a variable's variation is an interval.

We dwell on the idea of a moment in a variable's variation to introduce the idea of *rate of change at a moment*, meaning that a function has a rate of change that is essentially constant over a small interval of the function's independent variable. Since the rate of change is essentially constant over an interval, the change in the function over that interval is essentially equal to dy, where $dy = mdx$, as dx varies through that interval. It is with this image that we introduce the idea of a *rate of change function* r_f *for a function* f*,* meaning that every value of r_f gives the rate of change of f at a moment of f's independent variable. With the concept of rate of change functions, we are positioned to build a function whose values approximate values of f by accumulating changes in dy as x varies, starting from a reference point. We use the term *accumulation function* for functions that arise by their values having accumulated at some rate over small intervals of their independent variable.

It should be obvious that our approach entails developing integrals as accumulations from rate of change functions as the first major concept of the calculus. It is in this respect that we see the FTC as being at the core of the course from the outset. With this entry it is intuitively immediate that the rate of change of an accumulation function at any moment of its independent variable is the value at that moment of the rate of change function from which it is built.

The idea of integral becomes crystalized for students when we introduce the idea of a value of one function being *essentially equal to* the value of another—that making Δx so small that making it smaller produces no practical change in the estimate of the function's value. "Practical", of course, depends on context.

The second fundamental problem of calculus, knowing how much of a quantity you have at every moment and wanting to know how fast it is changing at every moment, entails reversing the process of creating accumulation functions from rate of change functions. The major insight that is required is to realize that any value of a function that gives an amount of a quantity at every moment must have accumulated at some rate over moments of the function's independent variable. That is, if $f(x)$ is an amount, then that amount accumulated from some reference point a, and therefore $f(x) = f(a) + \int_a^x r(t)dt$ for some rate of change function r. Put another way, the FTC becomes the *motive* for finding a method of deriving rate of change functions from accumulation functions.

This course evolved at Arizona State University over the past five years, and an electronic textbook for it now exists. The ideas have also been experimented with high school students in Israel. During the current academic year, a controlled experiment was carried out at ASU to compare students' learning in our and traditional approaches. In Fall 2015 two full-time faculty taught sections of Math 270T, traditional Calculus 1 (n = 180, 68) while one full-time faculty and one graduate student taught two sections of Math 270R, our revised Calculus 1 (n = 114, 35). The sections were undifferentiated in the schedule of courses so we believe that there was no selection bias among students. Thompson met with the instructors in summer 2015 to construct a 12-item pre-post test. All instructors agreed that the final set of questions addressed a broad spectrum of important understandings that students should have at the course's end. Students took the pretest in their first recitation meeting. The pretest was embedded in each instructor's final exam; thus, all students who took a 270 final exam took the pretest a second time.

Table 1 shows that there were no significant differences in pretest scores between students in 270R and 270T ($p < 0.23$) and a highly significant difference in their posttest scores ($p < 0.001$). Scheffe post-hoc tests showed no difference between traditional sections and no difference between revised sections, but each traditional-revised comparison showed a significant difference ($p < 0.001$). There were no significant differences among sections in terms of percent of students who passed the derivatives mastery test.

Table 1. MAT 270 Pre-post Comparisons

	PreTest		PostTest	
	Mean	StdDev	Mean	StdDev
Traditional	3.18	1.53	4.89	2.45
Revised	2.98		7.90	

Individual interviews of students in both treatments also showed distinct differences in the quality of their understandings. Also, students who dropped 270R did so largely because its emphasis on meaning and meaningful reasoning did not fit their expectations of a mathematics class. Star and Smith (2006) reported a similar result in the University of Michigan's implementation of Harvard Calculus. Addressing students' expectations in 270R will be an important goal in the future.

We close by pointing out that our meaning of differentials dy and dx, as changes in quantities that are related linearly, is at the heart of our approach. It is by establishing powerful *meanings* of constant rate of change, linearity, and differentials that we incorporate the FTC in deriving accumulation from rate of change and in deriving rate of change from accumulation.

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7. CURRICULUM DESIGN INCLUDING ASSESSMENT

Building and measuring mathematical sophistication in pre-service mathematics teachers

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We advocate that fostering mathematical sophistication should be a main role that advanced mathematics contents courses play in the university education of pre-service teachers.

Mathematical sophistication – a desired outcome of advanced mathematics courses

University mathematics teacher education programs face a fundamental problem of what Felix Klein (1924) called the Doppelte Diskontinuität (double discontinuity). The first discontinuity occurs with the transition from school to university mathematics, and the second discontinuity concerns whether this university education has the desired impact on their future work as mathematics teachers. (See Hefendehl-Hebeker (2013) for an overview of the problem and of contemporary efforts to tackle it). At issue with the second discontinuity is whether pre-service teachers are provided opportunities in their university coursework to learn the mathematics content knowledge (MCK), mathematics pedagogical content knowledge (PCK), and pedagogical knowledge (Shulman 1986) required for the work of teaching. Within the domain of PCK, Bass and Ball (2004) have identified and developed instruments to measure what they have termed *Mathematical Knowledge for Teaching*, which includes knowing which concepts best support students' understanding, and recognizing the nature of students' various conceptions and misconceptions. This knowledge is specific to the content of school mathematics, and is likely not to be fostered directly through advanced mathematics coursework. Therefore, an important issue within the domain of MCK is the role that advanced mathematics coursework has in developing the mathematics content knowledge that teachers actually need to teach school mathematics.

Szydlik and Seaman (2007) have identified specific aspects of MCK that are not content-specific, but rather knowledge of how to do mathematics, when they proposed the construct of *Mathematical Sophistication*. This construct refers to a person's mathematical behavior – the avenues of doing mathematics that one has at their disposal – and consists of an internalization of the values, behaviors, and habits of mind of the mathematical community that are powerful in learning new mathematics. The concept is rooted in a sociocultural perspective on mathematics learning (Bauersfeld 1979; Resnick 1989; Schoenfeld 1992): Through a process of enculturation in what it means to do mathematics, the learner aquires a mathematical point of view – thus "seeing the world in ways like mathematicians do" (Schoenfeld 1992).

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The development of the mathematical sophistication concept (along with a framework of norms characterizing it) by Seaman and Szydlik (2007) was motivated by their study in which a majority of pre-service elementary teachers were unable to use a teacher resource to make sense of an unfamiliar mathematical concept – a failure that the authors attributed to the participants not being able to think and act like mathematicians would have. This suggests that the pedagogically powerful forms of mathematical content knowledge intersect to a considerable extent with the forms of knowledge that allow mathematicians to create new mathematics. We agree with Seaman and Szydlik that building mathematical sophistication is critical not only for prospective research mathematicians, but for anyone engaged in mathematical learning – it should therefore be the main role that mathematics coursework plays in the preparation of teachers. While the mathematics content of advanced university mathematics courses might not have an obvious counterpart in school mathematics, an explicit goal of this coursework should be to allow students to acquire traits of mathematical behavior that empowers them to do and make sense of mathematics, and be able to enculturate these behaviors in their own future classrooms.

Building mathematical sophistication

Motivating students to value mathematical sophistication. A survey carried out by the first author (unpublished) indicates that a large portion of pre-service teachers is interested in university mathematics only as far as it is visibly related to their future jobs as teachers, rather than as an interesting scientific endeavor in and of itself. It is therefore important to convince students that mathematical sophistication is in fact useful – and in many situations even a requirement – for successful teaching. One approach in this direction are *interface activities* („Schnittstellenaktivitäten“) (Bauer und Partheil 2009; Bauer 2013a,b), which consist of specific homework problems ("Schnittstellenaufgaben") discussed in special recitation sections ("Schnittstellenübungen"), designed to establish connections between school mathematics and university mathematics – such as problems that highlight the use of advanced techniques from university mathematics in order to gain deeper insight into topics appearing in school mathematics (category C in Bauer 2013a and Bauer 2013b).

Designing advanced courses that help students gain mathematical sophistication. Mathematical behavior is a facet of mathematics knowledge that is rarely made explicit in mathematics content courses – perhaps it is often assumed that students will notice them implicitly. However, we argue:

(1) Mathematics content courses should make mathematical behavior more explicit.
(2) Mathematics content courses should involve students in more activities that require authentic mathematical behavior.

Here (1) entails showing avenues of knowing that the mathematical community has developed in general, but also "disclosing" the mental models and strategies used by the educator concerning the currently studied concepts and problems, respectively, in order to foster cognitive apprenticeship (Collins et al. 1989). The lecturing tradition in mathematics so far does not put much emphasis on these aspects – the focus is predominantly on the finished products (expressed in definitions, theorems and proofs) rather than on the acting mathematician's behavior. As for (2), reactions on the part of university educators might vary in a wide spectrum between the statements “We do this anyway” and “This is too difficult for the average student”. While it is true that challenging “Prove that …” problems can involve a variety of mathematical activities, it should be noticed that they do not cover the whole spectrum of mathematical behavior as conceptualized by the list of traits of mathematical sophistication from Seaman and Szydlik (2007). We argue that such activities are very well possible at every stage of mathematical education. (See also Bauer 2013c, where a case is made that this applies to school mathematics as well.) The example below, which involves the activities of *conjecturing* and *defining*, is given to support this point of view. We agree with Belnap and Parrot (2013) that conjecturing is a valuable mathematical activity for students, as it appears to involve many of the traits of mathematical behavior that Seaman and Szydlik as well as Schoenfeld (1992) identified. The example (see the box) shows an exercise problem from a course of the first author on Elementary Algebraic Geometry, which encourages experimenting with examples, verbalizing expectations, as well as stating and proving conjectures. Compare it to a version of type “Prove that for every curves of degree d, the intersection with a line …” – the same theorem is being proved, but the mathematical activities differ substantially.

Exercise: Intersection of plane algebraic curves with lines

(a) Find all lines that intersect the curve

$$\{(x, y) \in \mathbb{C}^2 \mid y^2 = x^3\}$$

in 1 point, 2 points, 3 points. (A picture of the curve might be useful to get some intuition.)

(b) Why can't there be more than 3 points of intersection?

(c) Suppose a curve $C \subset \mathbb{C}^2$ is given by a polynomial of degree d. Based on the previous example, what possibilities do you expect for the number of intersection points of C with lines? State a conjecture and prove it.

Measuring mathematical sophistication

Szydlik, Kuennen and Seaman (2009) developed a 25-item multiple-choice instrument that attempts to measure a student's level of mathematical sophistication with items designed for the following traits: 1) find and understand patterns, 2) classify and characterize objects based on structure, 3) make and test conjectures, 4) create models of mathematical objects, 5) value precise definitions, 6) value an understanding of why relationships make sense, 7) value logical arguments as sources of conviction, 8) have fine distinctions about language, and 9) value symbolic representations and notation. We were interested in answering the following questions:

1. What kind of adaptations are necessary for use of the items with German students?
2. Is this instrument, which was designed for use with elementary and middle school preservice teachers, also meaningful when used with pre-service Gymnasium teachers?
3. In which ways do beginning students show different mathematical sophistication than ending students (novice-expert comparison)?

As for 1), we found that few adaptations beyond mere language translation were necessary. (This should be seen in contrast with Delaney et al. 2008, where a number of changes accounting for cross-cultural differences were deemed necessary.) Preliminary results for 2) suggest that the items work well with pre-service Gymnasium teachers. This might be explained by the fact that the items are by design not bound to specific mathematical content, as they aim at measuring behavior that results from coursework rather than content that occurs in coursework. As for 3) we found in a subset of the items a significant difference between novices and experts (i.e., beginning and ending students), while little difference for a second subset. Further research is necessary in order to explain these findings – in particular it would be extremely interesting to uncover which of the findings can be attributed to the different nature of the chosen items (e.g. level of difficulty), and which to differing impact of university education on specific facets of mathematical sophistication.

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Courses in math education as bridge from school to university mathematics

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After describing the actual situation at the University of Hannover and our primary motivation for establishing a math education course as bridge from school to university mathematics we sketch a few basic ideas for designing the course. A crucial point is the intention to make the difference between school and university mathematics more explicit to first year students. This goal will be realized by making use of concepts and theory elements from research in math education that involve for example learning and epistemological analyses of mathematical topics in the transition from school to university. According to these ideas it seems important that the basic math courses remain essentially unchanged in form and substance.

Introduction

In most of the German universities students who want to become upper secondary teachers have to attend the same basic mathematics courses as students of pure mathematics, physics or other subjects. There are several reasons for this. On the one hand, in earlier times the focus was more on the subject than on the professional career, and the requirements w.r.t. mathematics were high also for the future teachers. So it was not uncommon that following a “Staatsexamen” a PhD thesis in mathematics was written. This idea still lives on, while the situation has become quite different in the meantime. The increasing number of high school pupils aiming for the “Abitur” requires an increasing number of mathematics teachers, and often these students are less focused on mathematics compared to those who took this career path in earlier decades.

With the introduction of the Bachelor/Master system in Germany, the traditional route towards becoming a teacher changed in most states. In Hannover, like in many other places, a polyvalent interdisciplinary Bachelor was introduced; it was designed to leave open both possibilities: that of a subsequent Master of Education for future teachers and that of a Master in the chosen major subject area. But trying to fulfill both needs at the same time is not optimal for both, in particular for mathematics where the courses build on each other and the students are familiarized with abstract concepts already in the first year. For many teachers to be – who often form the majority in the basic mathematics courses – it is a big problem to participate in courses that are primarily targeted towards future mathematicians (and physicists) without any accompanying reflection of the contents with respect to the topics treated in high school. Indeed, until this semester there was no course in the first year that took up explicitly and solely educational questions, i.e., mathematics teaching on its own.

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Also, the studies by (Pieper-Seier et al., 2002) confirm the impression held by many German mathematics departments: the different groups of students attending the same basic mathematics courses have very different attitudes towards mathematics; their findings make more precise where the differences are and how this may affect the students' success. Besides that it is known that the transition from school mathematics to university mathematics is problematic for most students. Main reasons are the "advanced mathematical thinking", and the different learning culture that is also related to new mathematical practices like giving proofs, see for an overview (Hoppenbrock et al. 2015). This leads to a situation where the potential of challenging math courses in the professionalization process of teachers is in general not realized. This problem also exists at the University Hannover regarding other science studies but also other philosophical studies. Therefore a joint proposal of nine different disciplines is taken up in the context of the German "Qualitätsoffensive Lehrerbildung".

Initiated by the regular reassessment of the study courses, the Faculty of Mathematics and Physics at Leibniz University Hannover decided to introduce a new course for students who want to become teachers in the first year that improves the situation from several points of view. The goal is to acquaint the students with math education ideas early on, and it should provide an explicit bridge between school mathematics and university mathematics.

Basic Ideas for Designing the Course

Overcoming Defensive Learning

We assume that the challenging contents and tasks in the first year courses Analysis and Linear Algebra show potentials for professionalization processes of students becoming math teachers. But often these potentials cannot effectively be realized. Instead students' learning is more or less dominated by the goal to pass the exams and shows facets of obstructed instead of motivated learning. Within Holzkamp's subjective scientific approach (Holzkamp, 1993) such a kind of learning is conceptualized as defensive learning in opposition to expansive learning. A prerequisite that is to some extent necessary but in general not sufficient for overcoming defensive learning is that students transform potential learning topics to actual learning topics on their own. According to the thematic and operational dimensions of learning topics this includes not only to intensify students' experiences of differences between their actual knowledge and possible knowledge but also to let qualitative differences become conscious and explicit. This means to strengthen students' self-awareness, for example in form of self-explanations during learning and doing mathematics.

Smoothing or Explicating Felix Klein's first Discontinuity?

In (Klein, 1933) Felix Klein proposed a course on school mathematics from a higher point that concludes the math courses and takes also into account the second discontinuity, i.e., the transition from university to school. Klein's course implies that students already have rather advanced math knowledge and can operate with this knowledge in a flexible and mathematically competent way, in particular that they do not only know facts and techniques but possess adequate, networked and critical ideas for validating and evaluating their significance. Nowadays, it might be doubted whether the average upper secondary teacher student shows this expertise after finishing the mathematical courses. This may be the main reason why such courses in the spirit of Klein are seldom proposed today.

Instead the typically proposed course or measure to support teacher students in building relations between university mathematics and school mathematics is established in the first or second semester and tries somehow to smooth out the ruptures between school and university (Hefendehl, 2013; Winslow et al., 2014). For reaching this goal there are principally two approaches at hand which may come mixed. A first one that adds aspects of the new university discourse slowly and step by step and a second one that develops university like problems starting with school mathematics. Following (Job et al., 2014) and massaging their arguments concerning the transition from calculus to analysis a little bit, smoothing the gap shows at least a tendency to blur the distinction between the different discourses which "tends to reinforce the empirical positivist attitude as an epistemological obstacle to learning" (ditto; p. 641). Additionally the smoothing-approach implies a massive intervention in the basic math courses which could be considered problematic since they are also attended by students of other studies like major mathematics or physics.

This does not mean that the existing math courses cannot and/or should not be optimized. But we intend to highlight the difference between the school and the university discourse and to clearly communicate that the latter does not constitute a completion of the first one and the former does not represent an essentially deficit discourse without relevant validations and justifications. We propose to explain and to accept the respective foundations and goals of these discourses and finally to provide conceptual tools for understanding students' own learning difficulties and ideas for their overcoming in the spirit of "expansive learning". This is the point where pedagogical content knowledge might play an important role.

The Role of Pedagogical Content Knowledge

The difficulties in the first year at university show on the one hand, that students mostly possess some willingness and motivation. But often they are (conscious-unconscious) ambivalent towards university mathematics. On the other hand the high dropout rates in mathematics (besides the students who leave the university or change their study one should not forget those students, who only pass the written exams without really mastering university mathematics) show that there are serious content related hurdles, since one should expect that ambivalence of willingness and motivation is a phenomenon in the first year of study independent of the field.

Our approach is grounded in the conviction that ideas and concepts from didactics can be helpful in clearing up students' own experiences and in developing strategies to overcome learning problems. Exemplarily we mention the praxeology concept in the Anthropological Theory of Didactics (Chevallard, 1999) that allows to characterize practices in their institutional existence and approaches in math education research for characterizing proofs (Boero, 1999), their cultural framing and differences to argumentation as such.

Making use of didactical concepts for self-explaining processes regarding personal learning experiences and emerging difficulties enforces to some extent a change in the perspective with respect to which those concepts are formulated and developed. This transformation is strongly related to something that is systematically conceptualized in the subject-scientific approach as reinterpretation or reconstruction of nomological theories in psychology, respectively the transition from the cause discourse, the discourse within which typically didactical theories are formulated, to the reasoning discourse, the discourse within which the psychological moment of human acting lives (Holzkamp, 1993). By introducing didactical concepts in view of the indicated change of perspective we also hope for positive learning effects with regard to didactical topics.

Outlook

In view of Shulman's (1986) fanning out of professional teacher competences the proposed bridging course focuses on content knowledge as part of the so called pedagogical content knowledge. The course should provide multifaceted mathematical knowledge that is helpful for explaining, representing and in particular validating mathematics in upper secondary school and aims at a more effective learning of basic higher mathematics. In terms of the Anthropological Theory of Didactics (Chevallard, 1999) it considers didactically reflected and deeply in mathematics grounded "technology" and "theory" aspects of mathematical practices, which typically do not become thematic in school and are so far usually only implicitly included in the first year mathematics courses at university. Moreover the course focuses on making available concepts and results from research in mathematics education, in particular on the transition from school to university, to support students regarding their self-understanding learning mathematics for becoming math teacher. We will accompany the introduction of the new course and its impact on students' learning by quantitative and qualitative oriented research. About the results we will report elsewhere.

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Designing examinations for first year students

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In my presentation I will discuss a new approach to the "examination problem" for first year students of mathematics. Characteristic is a competence oriented framework enabling valuable feedback for both students and instructors, as well as research in mathematics education at the university level. This approach has been developed and tested at the School of Education of the Technical University of Munich for teacher students of mathematics, but it is general enough to be adapted for bachelor of science students and many other studies involving mathematics.

An examination in its simplest form is, mathematically speaking, just a function returning a "passed" or "not passed" value, or a slightly more detailed grade on a linear scale. At a more sophisticated level, an examination can

- give a detailed and informative feedback for both students and instructors
- be a powerful instrument to change attitudes and habits of learning
- play an important role in course-design and determination of educational objectives
- provide custom-fit material for empirical studies and research

With respect to the transition problem, assessments in mathematics at the beginning university level are particularly challenging. One might speak of an "examination problem" for first year students. The traditional practice in mathematics gives an easy but unsatisfactory answer: Select some of the written homework exercises reflecting the scientifically most important contents of the course and use them in modified form for the final examination. The attitude is: "If and only if a student understood the lecture, then he or she will pass the exam." Inherent in this attitude is that the course is an isolated, self-contained construct, and that individual transformation processes are, in the end, of little interest. The method works well in identifying a group of "good students" (comprising about 25% – 33% of all students). Additional selection processes finally produce a group of students which are considered to be fit to do a PhD in mathematics and thus have a chance to become a researcher at the university or at a mathematical institute. This underpins an understanding of mathematics where "mathematician" is almost identical with "research mathematician". With a more comprehensive notion of becoming and being a mathematician inside a complex scientific and nonscientific human community, the examination problem is much more difficult and, from a didactical perspective, much more interesting. Various groups have to be considered: science students, teacher students for many kinds of different schools, engineers, computer scientists, etc. In the following, I will concentrate on teacher students at the secondary level, as this group is at the center of my work. Again, the traditional answer is easy:

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As there is only one mathematics, teacher students are not to be treated differently than science students.

In contrast to the traditional approach, a fundamentally redesigned and highly specific curriculum for teacher students of mathematics started in the Winter Term 2014/15 at the Technical University of Munich (I presented this at the precursor to this conference, the Oberwolfach meeting in December 2014). The mathematics courses of the first year now consist of two modules “Introduction to mathematics I/II” instead of the traditional distinction in “Analysis” and “Linear Algebra”. The courses are specifically designed for teacher students and include topics from Analysis and Linear Algebra as well as many other topics including Number Theory, Geometry, Graph Theory, and Set Theory. Methodically, individual transformation processes and the future profession as a teacher are central. Structurally, the courses have five main parts:

- Lecture (4 hours per week)
- Homework tutorial (2 hours per week)
- Teacher specific training (2 hours per week)
- Discussion and additional training (2 hours per week)
- Competence oriented examination

As with the other parts, the examinations have been redesigned and the new concept is currently evaluated. Two important aspects are:

(1) The examination is an integrative part of the course, not something which “has to be done at the end”. Students are informed at the very beginning about the structure of the examination; a mid-term examination is used to acquaint the students with the type of questions and the details of correcting and grading the answers; the results are discussed in class and individually using examples of answers as well as a detailed statistical analysis.

(2) The traditional mathematical examination model described above – which I would like to call the “linear model”, as it walks through the lecture content by content, is replaced by a “non linear model”: Instead of reflecting the development of the course from the first to the last lecture, items of different types are selected. The types themselves are given by the educational objectives of the course. Some examples of types are:

    - stating mathematical definitions
    - stating mathematical theorems
    - calculus and algorithms
    - visualization and diagrams
    - school mathematics from an advanced point of view
    - arguing and proving
    - reproducing or summarizing known proofs

In my talk, I will discuss the examination model and some of the types in more detail, together with a statistical analysis of correlations.

Students' perceptions of and conclusions from their first assessment experience at university

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As the perceived characteristics of assessment seem to have a considerable impact on students' approaches to learning, the way we assess has the potential to drive and to support students' learning. Therefore, this paper aims to give an insight into first year university students' perceptions of their first examination of an Analysis I-course and consequences which arise from these perceptions on the basis of interviews. First results show that students perceive "calculation tasks" to be dominant over proofs in this assessment, whereby some students start to learn from selectively chosen tasks which are perceived to be relevant for the exam. For a development of assessment tasks that support students' learning, the embedding of those tasks in the context of lecture and exercise sheets and especially students' perception of the tasks have to be taken into account.

Introduction

The perceived characteristics of assessment seem to have a considerable impact on students' approaches to learning (Miller & Parlett, 1974; Snyder 1971; Struyven, Dochy, & Janssens, 2003). Therefore, the way we assess has the potential to drive and to support students' learning (Brown, 2004; Gibbs & Simpson, 2004). In terms of the distinction between a surface approach (memorizing, reproducing etc.) and a deep approach (understanding, relating etc.) (Marton & Säljö, 1984; Entwistle & Entwistle, 1991; Entwistle & Ramsden, 1983), impacts of different assessment types on students' approaches to learning have been investigated. Thereby it seems to be easy to provoke a surface approach and hard to encourage a deep approach. Entwistle & Entwistle (1991) found that multiple-choice formats provoke a surface approach, while open, essay-type questions tend to encourage a deep approach. Generally, assessment methods which are perceived to be inappropriate ones are likely to provoke a surface approach (Struyven, Dochy & Janssens, 2005). However, it has to be considered that these findings do not stem from the context of mathematics at university and it is not clear to what extend they apply in different contexts (Joughin, 2010). In the context of mathematics it seems that oral assessments can encourage a deep approach (Iannone & Simpson, 2014).

In a survey of assessment in UK mathematics departments Iannone and Simpson (2011) found that assessment is (at least in UK) dominated by closed-book examination. Besides, students perceive closed-book examinations to be the best discriminator of mathematical ability although they perceive assessment of memory to be dominant over assessment of understanding for closed-book examinations (Iannone & Simpson, 2013).

In addition to the outer form of the assessment, students' perceptions and expectations on which skills, contents and concrete tasks will be assessed are likely to influence students'

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approaches to learning. Smith et al. (1996) developed a taxonomy of skills needed to complete a given mathematical task with the categories (A1) *factual knowledge and fact systems,* (A2) *comprehension,* (A3) *routine use of procedures,* (B1) *information transfer,* (B2) *application in new situations,* (C1) *justifying and interpreting,* (C2) *implications, conjectures, and comparisons,* and (C3) *evaluation*, mainly to assist lecturers in writing examination questions. They found that most of the examination papers they analyzed were biased towards Group A tasks (see also Ball et al., 1998). Analyzing questions of examinations between 2006 and 2012, Darlington (2014) found that about one half (54.1 %) of the average number of marks available for each examination needed Group C skills, while 33.6 % could be completed with Group A skills.

Using a framework of Lithner (2008), Bergqvist (2007) analyzed the 212 tasks collected from all introductory calculus courses offered at four different Swedish universities during the academic year of 2003/2004. She found that about 70 % of the tasks were solvable by imitative reasoning, i.e. by copying algorithms or recalling facts. Tallman et al. (2016) coded 3735 exam items of 150 Calculus I final exams (using a framework they developed) and found that 78.7 % of the items required students to recall and apply a rehearsed procedure, while only 14.72 % of the exam items required students to demonstrate an understanding of an idea or procedure. Interestingly, 68.18 % of all instructors who submitted exams indicated that they frequently require their students to explain their thinking on exams.

Such frameworks have several limitations. Answering one particular question may involve more than one skill and may call on different skills from different students (Darlington, 2014). Also, the categorization of a task might depend on its embedment in the context of lecture and previously practiced tasks, especially with regard to categories which include terms like “routine”, “new” or “known”. A seemingly routine procedure task may be novel and challenging for students who don’t know the procedure and tasks that might appear to require an understanding of a particular concept are amenable to being proceduralized (Tallman et al., 2016). Finally, the skills necessary to complete a task might be interpreted differently by different persons. One example is that the interpretation of Tallman et al. (2016) does not align with the instructors perceptions relative to the extent to which students are required to explain their thinking.

With regard to the potential of assessment to support students’ learning, students’ interpretation of the examination tasks is of particular interest. Therefore this paper aims to give a first answer to the following questions:

- How do students experience their first examinations at university?
- Which consequences arise from these experiences?

Methodology

To investigate these questions interview data of eight students which attended their second semester at university at the time of the interview were analyzed. The sample comprises five female preservice teachers, two (one female, one male) students heading for a bachelor’s degree in mathematics and one female student heading for a bachelor’s degree in physics. All of them had participated in a (definition-theorem-proof-based) Analysis I -

course and the written closed book exam at the end, which has been their first or second examination in mathematics at university. To pass the module it was required to achieve at least 50 percent of the attainable points of weekly exercise sheets as well as passing the exam.

The analysis of the data did not follow any systematic methodology. Parts with direct reference to the questions were excerpted and compared. Therefore, the first results given below should be seen as examples only.

The participation in the interviews was voluntary and not compensated. The interviews were led, audio taped and transcribed in German language. The parts being cited in this paper were translated by the author.

The Exam

I try to give a short description of the Analysis I exam using the categories of Smith et al. (1996): About 1/6 of the achievable points could be reached with *"factual knowledge and fact systems"* (A1), such as stating a definition, or a necessary and non-sufficient condition for the convergence of a sequence of real numbers. About 1/5 of the points could be reached by (A2) *"comprehension"*, such as deciding whether a given statement is true or false. About ¼ of the points could be reached by solving tasks classified as (A3) *"routine use of procedures"*, such as determining limits of sequences, radii of convergence of power series, or Taylor polynomials of functions. Thus, about 62 % of the points could be attained with Group A skills. Group B skills were needed for 15 % of the points, which come from tasks categorized as (B1) *"information transfer"*, such as finding the derivative of a piecewise defined function. The remaining 23 % of the points belong to Group C tasks, with 4 % for (C1) *"justifying and interpreting"* (e.g. the composition of two injective functions is injective) and 19 % for (C2) *"implications, conjectures, and comparisons"* (e.g. comparison of definitions, construction of examples or counterexamples).

I would like to stress that for many of the tasks, the categorization is not obvious and unambiguous, due to the above mentioned limitations of such frameworks.

First Results

To have a first insight into students' perceptions of the assessment tasks, we look at the following excerpt.

> *But then I have seen the first examination. And it was composed originally of the tasks we have done on the exercise sheets. There was nothing additional. Well, maybe sometimes there was a function defined differently or so. Right, but it was original the exercise sheets, and the definitions. There didn't appear anything else. And a simple proof, but it was also in the exercises, we did before, at least similar. And then I thought to myself, I will learn only from the exercises.*

Here, the assessment tasks are perceived to be very close to the tasks of the exercise sheets. A consequence of this perception is to focus on the exercise tasks when preparing for the exam. One step further goes the next excerpt, which distinguishes those tasks of the exercise sheets to be relevant for the assessment, which are "calculation tasks".

> *Well, for me, for example in Analysis the lecture didn't really have a lot in common*

> *with the exam. Therefore I said to myself this year, I only work on by myself or try to understand those tasks of the exercise sheets which are relevant for the exam, thus all calculation tasks, such as derivatives and such things. Because all those proof tasks, nary of them appeared (in the exam). And I was a bit angry, because I was stressed out the entire half year: "Oh god, I cannot solve the exercise sheets, I will flunk the exam", and in the end it was only calculations, what we needed to know.*

The opinion that especially the "calculation tasks" of the exercise sheets are relevant for the assessment, while proofs would appear hardly and only on a simple level could be found by every interviewee. As a consequence of this perception some students' tend to be engaged in selectively chosen tasks which are perceived to be "relevant for the exam, thus all calculation tasks". Some students start to completely omit proofs. On the other hand, on the basis of the experience that the assessment is feasible a decline of stress and increase of motivation could be observed by some students in the second semester.

> *I feel good about my study because I passed the examinations. And the knowledge that complex proofs and such things do not appear in the exam is good for the exercise sheets too. However, I try to do the proofs of the exercise sheets, but I know I can pass the exam nevertheless. In fact I had heard before that they cannot pick the really difficult proofs for the exam. But now, when I really saw it, it is a kind of emotional motivation.*

Discussion

It seems that students distinguish essentially two categories of tasks, namely "calculation tasks" and "proof tasks". It is not clear how these categories relate to the categories of Smith et al. (1996) or categories of the other authors mentioned above. However, it seems likely that there is an overlap of the categories "calculation tasks" and "routine use of procedures" (A3), as well as of the categories "proof tasks" and "justifying and interpreting" (C1) or generally Group C respectively. According to the categorization above, one forth the points could be achieved by "routine use of procedures", which is indeed the highest scoring category, but this categorization does not align with the perception that "it was only calculations, what we needed to know".

The opinion that especially the "calculation tasks" of the exercise sheets are relevant for the assessment, while "proof tasks" would appear hardly and only on a simple level could be found by every interviewee. An explanation might be that the proportion of "calculation tasks" in the exam was higher than it was on the exercise sheets. Anyway, the interviewees seemed to have had expected different tasks in the assessment. This shows how important the embedding of the assessment tasks in the context of the lecture and the exercise sheets is, especially, if we are interested in consequences of the assessment tasks on students' approaches to learning.

It is worth mentioning, that primarily the tasks of the exercise sheets were experienced to be relevant for the examination by the interviewees, while the actual content of the lecture took a back seat: According to the first excerpt, the examination "was composed originally of the tasks we have done on the exercise sheets", while "the lecture didn't really have a lot in common with the exam", as the second excerpt puts it.

The consequences of students' experiences of the examination are straightforward: Preparing for an examination, students plan to focus on the tasks of the exercise sheets, especially those tasks, which are perceived to be relevant for the examination, thus the "calculation tasks". This tends to end in a selective learning of some separate tasks by some students, which is a rather surface approach to learning. Such an approach is unlikely to provide a proper understanding of the lecture content. Furthermore, it might affect negatively students' motivation and confidence (Göller, 2015). According to Struyven et al. (2003) a turn towards a surface approach is to expect if students perceive the assessment tasks to be inappropriate.

To evaluate students' actions, it seems to be necessary to take their emotional condition into account. It seems that being "stressed out the entire half year: 'Oh god, I cannot solve the exercise sheets, I will flunk the exam'" is a predominant feeling of many students. In this respect, the drift towards a surface approach doesn't necessarily have to be seen as a fundamental attribution, but rather as the only way to cope with the requirements. The fact that the interviewees seemed to have had expected different tasks points to difficulties of first semester students to define their learning goals. So, the examination defines what is important and what is less important. This clarification of the learning goals may imply a decline of stress and increase of motivation for some students, as the third excerpt shows.

We have seen how the assessment influences students' approaches to learning. Thus, a proper choice of examination tasks has the potential to drive students' learning towards a deep approach. Besides the assessment tasks, the embedding of those tasks in the context of lecture and exercise sheets and especially students' perception of the tasks play an important role for students' approaches. Therefore, a good alignment of lecture, exercise sheets and assessment, as well as a better understanding of students' perceptions of assessment, tasks, lecture content, and their interplay seem to be crucial for the development of such tasks.

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Fit for the job – The expertise of high school teachers and how they develop relevant competences in mathematical seminars

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In this report we present our teaching concept for mathematical seminars. The concept is geared towards helping teachers in training to increase their reading competence. This is achieved by training them to overcome difficulties in understanding through relevant questions and to design their own exercises. The topics of the seminar are chosen to be suitable for high school students. We also report on our first observations and share some results obtained during the pilot phase of the seminar (summer term 2015).

Theoretical aspects and motivation

High school teachers in training want two things: "We would like courses specifically designed for us." and "We would like courses that add value to our future job as teachers."

Lecturers mostly agree with the second wish and are confronted with the problem to decide what constitutes the specific knowledge a teacher must acquire. In order to give a (partial) answer to this question we use the categories introduced by Shulman (1986, 1987), who classifies teacher knowledge by distinguishing content knowledge, pedagogical content knowledge, pedagogical knowledge and curricular knowledge.

In German teacher training programs pedagogical and curricular knowledge is addressed in specific courses. Thus they do not play an important role in the content oriented mathematics courses. Therefore we will concentrate on content and pedagogical content knowledge.

The importance of content knowledge was underlined as follows by Ball/Lubienski/Mewborn (2001, p. 440): "The assertion that teachers' own knowledge of mathematics is an important resource for teaching is so obvious as to be trivial."

Shulman (1986, p. 9) describes pedagogical content knowledge as "the particular form of content knowledge that embodies the aspects of content most germane to its teachability". As a part of pedagogical content knowledge he counts knowledge of "the ways of representing and formulating the subject that makes it comprehensible to others" and furthermore "an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons. If those preconceptions are misconceptions, which they so often are, teachers need knowledge of the strategies most likely to be fruitful in reorganizing the understanding of learners".

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We consider it important to point out that teachers also need tools to deal with *their own* pre- or maybe misconceptions before they can competently judge the level of understanding of learners. Teachers also need to be able to familiarize themselves with new mathematical contents and decide in what form the material is suited for classroom presentation. This is closely related to the teachers' competency in reading mathematical texts, which is important for their training in itself. Furthermore, teachers also have to design worksheets and exercises for their students in such a way that the material

- really helps their students to understand the content,
- provides helpful exercises and
- allows for the students to assess their progress.

At German universities, courses for teachers in training tend not to offer much purposeful practice in the above-named skills. Rather it is assumed that certain competences in reading and the necessary skills for dealing with misconceptions are automatic by-products of the mathematical training.

Design of the course

Since the summer term 2015, we have been offering an additional seminar at Paderborn University that addresses the two student demands mentioned above and offers training in some of the relevant competences.

German universities in general offer two different types of courses: lectures and seminars. In a seminar each participant is assigned a topic on which he or she works individually. Students are supposed to prepare a 90 minute talk in advance. During the term, instruction is centered around these talks.

Our concept basically works with the seminar structure that we combined with extra consultation elements. Prior to the start of the term an initial workshop is held to clarify organizational matters and introduce students to the method of reading with stumbling blocks (Hilgert/Hoffmann/Panse 2015). Here each student can choose his or her individual topic for the 90 minute talk. It is very important for us that the topics are closely related to school curricula.

Throughout the semester, the participants work on their talks and on an elaborated written version of their topic. During this process they are supported in obligatory biweekly meetings with the lecturer, who helps them to structure the work by giving them detailed work assignments. Apart from the mathematical content, the elaborated written version has to contain contents a worksheet for pupils and a quiz with yes/no-items. For extra guidance, students are offered a consultation with their lecturer once a week. Towards the end of term the real seminar takes place, during which every participant gives his or her talk.

Research questions

We investigate the seminar concept in the context of "Design-Based Research" and we also consider the following questions: "What are the difficulties future teachers encounter when they read mathematical texts?", "To what extent is it possible to influence the reading behavior of the teachers in training?", "What are their main difficulties in designing effective mathematical exercises?", "To what extent is it possible to increase their skills in designing mathematical exercises?", and "What are the difficulties they face when preparing and giving talks?", "To what extent is it possible to help them increase their skills in preparing and giving talks?"

Observations

- In the pilot phase of our study we conducted interviews with students who actually gave presentations. We could not get hold of the ones who dropped out during the preparatory work. Concerning their individual reading routines no great changes were reported. This is not surprising, since those students had already a minimum of 2 years study experience and had already developed successful individual reading routines, often similar to the stumbling block method.
- We would like to mention that students who discontinued the seminar did so after they were asked to precisely name the difficulties they had in reading the mathematical text and to formulate concrete questions for a discussion with their supervisor.
- With regard to the creating of mathematical exercises and yes/no-quizzes it turned out that students did not prepare those in the way intended by the lectures. Instead of basing them on stumbling blocks and difficulties they had encountered themselves, the students said they had searched the internet. Interestingly, however, they realized that subconsciously they had chosen exercises that were related to their own stumbling blocks.
- With regard to the given talks we would like to point out that although it is true that almost every talk can be improved, the participant's talks were never less than satisfactory, many of them were very good. A typical weakness of the presented talks was an emphasis on mathematical trivialities while more subtle points were glossed over.

  Participants received a lot of constructive and helpful feedback for instance concerning their roles as teachers and their classroom presence, but in the interviews they seemed to focus on feedback on technicalities like the use of the black board.
- Finally we want to say that the students found the seminar very helpful and emphasized that this is a valuable course for their future job as teachers.

Outlook

We placed two competencies in the center of our attention: Being able to set up adequate exercise sheets and quizzes as well as being able to adequately present mathematical content. In both respects specific deficits were identified, and the obligatory creation of stumbling blocks shows potential to deal with them. The next round of this design-based research will have its focus on these deficits as well as methods of improvement.

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Mathematics students' perceptions of summative assessment: the role of epistemic beliefs

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The assessment diet of mathematics students in the UK is very uniform, with closed book examination being by far the most used method of assessment and oral assessment being largely absent. This picture sits against the backdrop of calls in the general higher education literature to move away from traditional assessment towards innovations to facilitate the shift away from a testing culture and better prepare students for the workplace. In this paper I will report findings from projects aiming at investigating mathematics students' perceptions of summative assessment and some early indications of the role that epistemic beliefs have in shaping these perceptions. I will then explore the significance of these findings for mathematics assessment and curriculum design.

Introduction: Students' perceptions of summative assessment

Marton and Saljo (1997) were amongst the first researchers to show a close connection between what students perceive summative assessment to require and the way in which they engage with their subject of study. This is to say that if students perceive an item of summative assessment to require understanding, they will try to engage with the subject in a way that fosters understanding, or with understanding as their goal. Likewise if they perceive assessment to require just memorization of facts they will engage with the subject at a surface level. Although this relationship is not always straightforward and changing assessment type does not always lead to a change in the way in which students engage with learning (Baeten, Dochy and Struyven, 2008) it is nevertheless important to investigate undergraduate students' perceptions of summative assessment.

Indeed there is a large body of higher education literature that does just this (Scouller, 1998; Birenbaum, 2007, just to cite two examples) and a comprehensive review article by Struyven et al. (2005) summarises just what these perceptions are. The picture emerging is that of a student body disenchanted with the role of traditional assessment which they perceive as hoops to jump in order to obtain marks and detrimental to their learning. The students on the other hand praise the learning opportunities afforded by what they call innovative assessment believing this is better at testing capabilities in their subject and preparing them for the workplace. This position resonates with much of the literature related to summative assessment and assessment policy (Medland, 2014) which calls for a radical change in the way in which students are assessed. Indeed it is often suggested that summative assessment should move towards *assessment for learning* and leave the current testing culture behind (Brown, 2004).

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First study: Mathematics students in the UK

A close look at the literature mentioned above and a meta-analysis of the studies reviewed in Struyven et al. (2005) reveals that there is a strong sample bias towards the social sciences (mostly education and psychology) and medicine in the studies included in the review (Iannone and Simpson, 2014). Indeed only a very small study (with 6 participants) involved physics students, and none involved mathematics students. Therefore we decided to replicate methods used in the general literature to find out mathematics students' perceptions of summative assessment. We adopted a mixed methods design with a survey (N = 148) aiming at ascertain how students wish to be assessed and what methods they perceive to be best discriminator for academic ability, followed by semi-structured interviews (12 students) with volunteering students. This study was carried out in two high-ranking research-intensive universities in the UK. A complete description of the study and its findings is in Iannone and Simpson (2014) but here we focus on two findings: 1) mathematics students prefer to be assessed by assessment methods they perceive to be good at discriminating for academic ability, and 2) mathematics students believe that the best assessment method to discriminate for academic ability and the fairer method is the closed book examination (with some caveats). Mathematics students would like, however, to see a little more variety in the way in which they are assessed when the whole assessment pattern across the 3 years is considered. These findings are in contrast with what is reported in the general literature and the question arises as to whether the context (the discipline and the instructional context) is playing a role in shaping these perceptions. Indeed in the qualitative part of the study students often refer to what they perceive mathematics to be when they explain their assessment preferences:

> *I assume if they're doing really well in the exam [...] if you're going to get a really high mark, it's being able to really understand it, because they could throw any question at you and you have to be able to apply the knowledge to that question, [...] So I think if someone's doing really well in maths exams, they're actually just got really, really good understanding.* (Tina)

> *Maths isn't really the sort of thing you put into your own words, like an arts, or a social science.* (Tanja)

We then made the hypothesis that the way in which students perceive their discipline could be an influencing factor on their perceptions of assessment, and that the reason why our findings are in contrast with much of the general literature is that the voice of the students in the hard/pure sciences (in the sense of Biglan, 1973) has not been heard. In order investigate whether the way in which students perceive their discipline (their discipline-based epistemic beliefs) is an influence on students' perceptions of summative assessment we replicated the study in the same institutions but with education students.

Second study: Education students in the UK

In order to test out hypothesis we carried out a second study (Iannone and Simpson, submitted) in the same institutions, but with education students. We choose education as this discipline is soft/applied in Biglan's classification – the opposite of mathematics. We selected the same higher education institutions to keep some of the context similar. We used the same tools and design of the previous study with the same survey (N = 57) followed by semi-structured interviews (6 students). Note here that the cohorts of education degrees in the two participant institutions are much smaller than that of mathematics degrees and this partly accounts for the difference size of the two data sets. The results were much more in line with what is suggested by the general education literature. We found that 1) also education students prefer to be assessed by assessment methods which they perceive to be good discriminator of academic ability and 2) they perceive closed book examinations (and generally traditional assessment methods) to be inadequate to assess the academic capabilities needed for educations and prefer to be assessed by innovative methods. The reason the students give in the qualitative part of the study are again linked to their own understanding of what constitutes knowledge in education:

> *So I think academically, exams don't actually really show the person's own intelligence. It's just how much you can remember something on a piece of paper, whereas coursework you can get out back and research, find your own personal experience [...] So I think academically ... academically exam shows off the person's memory, but coursework shows off the personal input and things like that.* *(Mary)*

> *... it is quite easy to kind of just memorise it for the exam and then, two weeks later you have probably forgotten most of it. But you know what you're going to write, you. ... it's almost more formulaic, while an essay is much more ... you can put much more of your personal style into it because you've got the time.* *(Georgiana)*

The results of the two studies seem to indicate that the students' preferences and perceptions of summative assessment are influenced by the way in which they perceive their subject and what they think constitute knowledge in their subject. This finding has important implications for the way in which we design curriculum and assessment in mathematics.

Discussion and concluding remarks: back to mathematics students and assessment for learning

I have discussed at the start of this short paper how the general literature strongly advises to move towards *assessment for learning.* Birembaum et al. (2006) highlight the need to shift from assessment of learning, characterised by a feed out function when summative, to assessment for learning, characterised by a feed in function and rich feedback. Such assessment for learning is also *authentic*, namely it connects to situations that could arise in the workplace. This is a particularly relevant dimension for the UK, where the employability discourse is dominating higher education debates at the moment. While it is clearly desirable to offer mathematics students rich feedback and closed book exams is not a type of summative assessment which facilitate rich feedback (in many universities in the UK students are not allowed to see their exam papers and only receive a mark at the end of the summer term), this shift cannot be conducted without keeping students' perceptions and preferences in mind. Curriculum designers and assessment setters need to take into account students' preferences and their desire to see a little more variety to their assessment diet. Recently in the UK some assessment variety has been (albeit slowly) introduced. Most degree courses in mathematics now have a compulsory project in their third year (with or without a presentation component) and are introducing problem-solving modules in their first year – which are typically assessed by coursework only. These assessment methods have the potential to be rich in feedback and, if paired with oral presentations, could offer mathematics students the opportunity to practice communication and oral skills.

The direction of future research is two-fold. On the one hand, given that students epistemic beliefs play an important role in their perceptions of assessment, we ought to investigate how these beliefs are formed and how to best measure them. Qualitative methods (as started in Perry's work, 1970) are appropriate and accurate but much time consuming; tools adopted for quantitative methods are still under debate and most instruments available have been subject close scrutiny and some criticism (DeBacker et al. 2008).

On the other hand the introduction of new assessment methods should be evaluate not only for validity and reliability, but also to monitor the impact that the introduction of the new assessment has on mathematics students' perceptions and engagement with learning. While it is certainly necessary to re-think the way in which we assess students, if only to introduce a little more variety in their assessment diet and think of suitable methods to introduce assessment for learning in university mathematics, this cannot be done without listening carefully to the students' voices.

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Pre-service teachers' abilities in constructing different kinds of proofs

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The University of Paderborn offers the course "Introduction into the culture of mathematics" for first-year pre-service teachers to help them transition to higher mathematics, especially to deal with mathematical proofs. This course is framed by different studies to evaluate and to refine it in a design-based research scenario. During the fourth offering of the course in 2014/15, we investigated students' benefits concerning proof competencies, acceptance and understanding in detail. In this contribution, I will discuss students' proof productions at the end of the course.

Introduction

The University of Paderborn offers the course "Introduction into the culture of mathematics" to ease students' transition to the tertiary level. This inquiry-based transition-to-proof course has been developed by Rolf Biehler and was held for the first time in 2011/2012 as a requirement for the first-year secondary (non grammar schools) pre-service teachers. During this course, the students explore mathematical issues (e.g., figurate numbers) and learn to construct generic proofs and formal proofs. During the fourth offering of the course in 2014/15, we investigated students' benefits concerning proof competencies, acceptance, understanding, self-efficacy and beliefs in detail. In this contribution, I will point out some findings concerning students' proof construction at the end of the course. The analysis is a part of a larger empirical study that forms the core of the Ph.D. project of the author.

Theoretical Background

In the field of mathematics education, different kinds of proofs have been introduced and discussed by educators and mathematicians (Dreyfus et al., 2012). The concept of generic proofs has become an especially prominent pedagogical tool at the secondary and tertiary level (e.g., Rowland, 2002; Stylianides, 2010). Here, I refer to the concept of generic proof that has been developed by Kempen and Biehler (2015): A generic proof consists of a generic argument illustrated by concrete examples and a written argumentation about its validity and generality.

Additionally, different (geometrical) representations are said to be useful both for constructing and understanding mathematics proofs (e.g., Flores, 2002). Following the recommendations from the literature, we made use of four different kinds of proofs in our course: the generic proof with numbers, the generic proof in the context of figurate numbers, the proof in the context of figurate numbers making use of "geometric variables" and the so-called formal proof (see Kempen & Biehler, 2015).

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Research Questions

Despite the prominent status of the different kinds proofs mentioned above, learners' ability to construct these kinds of proofs on their own has not yet been investigated in detail. The research questions are: (1) How do the students formulate arguments at the end of the course, when they are asked to construct the four different kinds of proofs mentioned above? (2) Do the students succeed in constructing general valid verifications when using the different kinds of proofs?

Methodology

We investigated students' abilities in constructing the different kinds of proofs in the final exam of the course. There, the students had to construct the four different kinds of proofs to prove one single statement (see below.).

Task and possible solutions

The statement to be proven is: "The sum of six consecutive natural numbers is always odd". Possible solutions for constructing the four different kinds of proofs are shown below.

Generic proof with numbers:

$1 + 2 + 3 + 4 + 5 + 6 = 21$ is an odd number; $4 + 5 + 6 + 7 + 8 + 9 = 39$ is an odd number

In every sum of six consecutive natural numbers, you will always have (exactly) three odd and three even numbers. The sum of the three odd numbers will always be an odd number. After adding the three even numbers, the result will always still be an odd number.

Formal proof:

Let n be a natural number. We have: $n + (n+1) + (n+2) + (n+3) + (n+4) + (n+5) = 6n+15 = 2q + 1$, where $q := 3n + 7 \in \mathbb{N}$. So by definition, the result is an odd number.

Generic proof in the context of figurate numbers:

In the representation of the sum of six consecutive natural numbers by figurate numbers, one always obtains the same shape of stairs on the right side. By transforming these stairs (taking the edge at the bottom right and putting it above) one always obtains two equal parts. But after this division by two, one always obtains the remainder three. So the sum will always be an odd number.

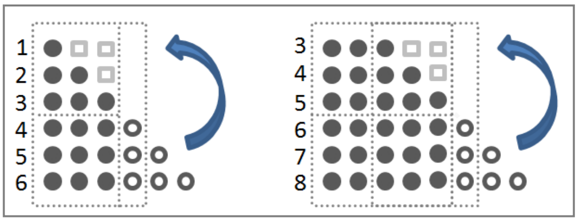


Fig. 1: The sum of six consecutive numbers represented by figurate numbers.

Proof with geometric variables

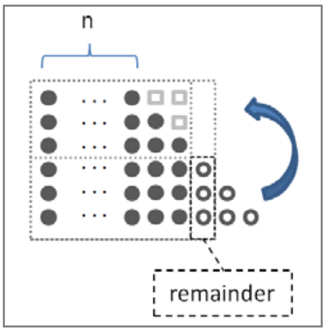


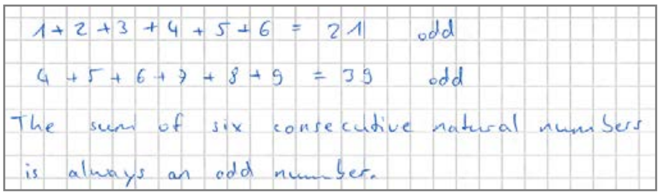
Fig. 2: A proof with "geometric variables" and figurate numbers.

Set of categories

In order to analyze and compare the proofs produced, a set of categories was needed that could be applied to all four kind of proofs and to any proofs produced. We used a modified version of the set of categories presented in Kempen and Biehler (2014) to accomplish these needs. Since it is not possible, to give concrete examples here for every category concerning all for kind of proofs, I will only clarify the categories concerning the generic proof with numbers. However, addition information about the categories concerning the other kinds of proofs will be given.

(1) n. p.: not processed

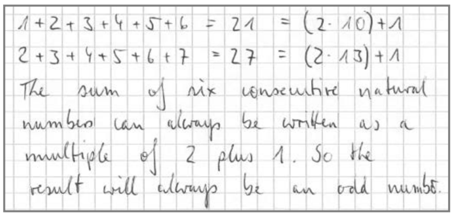
(2) empirical: The truth of the statement is inferred from a subset of (concrete) examples. [See figure 3.]



$1+2+3+4+5+6 = 21$ odd
$4+5+6+7+8+9 = 39$ odd
The sum of six consecutive natural numbers is always an odd number.

Fig. 3: A student answer, which belongs to the category "empirical".

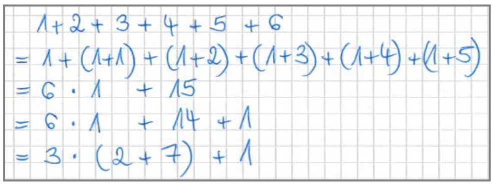
(3) pseudo: The answer is given by merely stating or paraphrasing the statement that the sum is always odd/ wrong solutions/ irrelevant information/ construction of figurate numbers without a geometrical or useable arrangement. [See figure 4.]



$1+2+3+4+5+6 = 21 = (2\cdot 10)+1$
$2+3+4+5+6+7 = 27 = (2\cdot 13)+1$
The sum of six consecutive natural numbers can always be written as a multiple of 2 plus 1. So the result will always be an odd numb.

Fig. 4: A student answer, which belongs to the category "pseudo".

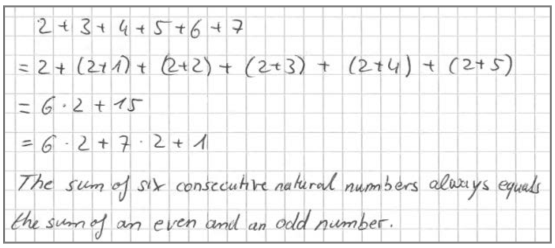
(4) fragmentary: only fragmentary information are given/ meaningful arrangement of figurate numbers without further information.
[See figure 5.]



$1+2+3+4+5+6$
$= 1+(1+1)+(1+2)+(1+3)+(1+4)+(1+5)$
$= 6\cdot 1 + 15$
$= 6\cdot 1 + 14 + 1$
$= 3\cdot(2+7)+1$

Fig. 5: A student answer, which belongs to the category "fragmentary".

(5) argumentation with gap: The students derives the conclusion by a connected argument and from generally agreed facts of principles. Just because of (minor) inaccuracies the explanation is not a complete verification. [See figure 6.]



$2+3+4+5+6+7$
$= 2+(2+1)+(2+2)+(2+3)+(2+4)+(2+5)$
$= 6\cdot 2+15$
$= 6\cdot 2+7\cdot 2+1$
The sum of six consecutive natural numbers always equals the sum of an even and an odd number.

Fig. 6: A student answer, which belongs to the category "argumentation with gap".

(6) complete explanation: The student derives the conclusion by a connected argument and from generally agreed facts of principles without doing (formal) mistakes. [Examples of complete proof productions have been given above.]

Results

The results concerning students' abilities in proof construction are shown in figure 7. After attending our course, 58% of the students succeed in constructing a complete generic proof with numbers. Overall 82% ("arg. with gap" + "compl. expl.") construct a meaningful reasoning. Concerning the formal proof, 44% accomplish this proof without (formal) mistakes. And meaningful attempts are given in 84% ("arg. with gap" + "compl. expl."). To construct the proofs in the context of figurate numbers seems to be harder for our students. Only 20% succeed in constructing the generic proof with figurate numbers and only 38% in the case of geometric variables.

In the context of figurate numbers, the percentage of answers belonging to the categories "pseudo" and "fragmentary" are astonishing. It seems that the work in this notational system is a hurdle for our first-year students.

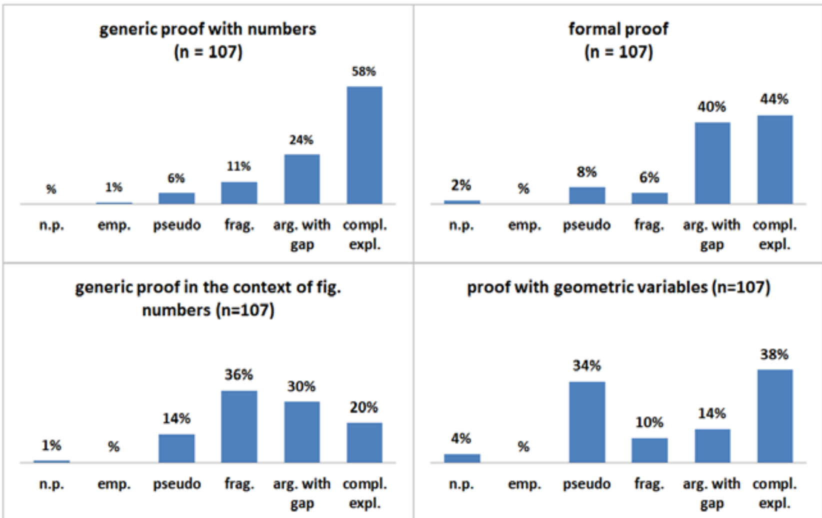


Fig. 7: Results concerning proof construction from the final exam of the course.

Final Remarks

In this contribution, I presented some results concerning students' abilities in constructing different kind of proofs. After attending our course, the majority of our pre-service teachers were able to use the symbolic mathematical language to construct a formal proof for a theorem of elementary number theory. Most of them were able to construct a generic proof with numbers. But concerning the use of the notational system of figurate numbers, students' responses display various difficulties. Here, the high percentage of the category "pseudo" illustrates students' problems in using figurate numbers as a (geometrical) representation. It becomes clear that the use of figurate number does not necessarily support the learning of proofs. Its use and interpretation has to be considered as an additional learning subject.

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Relating content knowledge and pedagogical content knowledge in the mathematics teacher education

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In the mathematics teacher education the various dimensions of professional competencies should be treated in an integrated way to help the students to build up their own network of content, pedagogical content, and general pedagogical knowledge. At Humboldt-Universität zu Berlin we introduced courses that explicitly integrate content knowledge with pedagogical content knowledge. In our presentation, we will illustrate two examples of such courses, namely the course "Stochastics and its didactics" as well as "Algebra/number theory and its didactics".

The general situation

The professional competencies of mathematics teachers should be characterized by an entity of content knowledge, pedagogical content knowledge, and general pedagogical knowledge. Our experience documents that teacher students, in general, are not able to acquire this entity of knowledge by themselves during their university studies. An indicator for this is, for example, the conclusion of students that the mathematics taught at universities is irrelevant for their later teaching at schools. In contrast to that the students overemphasize their personal role as teachers for the learning success of their perspective high school students. To summarize, we are here confronted with "Klein's double discontinuity" in the mathematics teacher education: Beginner students perceive the mathematics taught at universities as totally different from the mathematics that they are planned to teach at schools after completion of their university studies. In fact, they have serious doubts that the mathematical knowledge acquired at the university has any impact for their future teaching at school. These doubts are to some extent justified, if the instruction of the mathematical content knowledge at the university remains isolated and does not become interconnected with the other dimensions of the students' future professional knowledge. Moreover, most students will primarily focus on the personality traits of a successful teacher and thus orient themselves just on the mathematical knowledge, which is needed at school. Therefore, an important task of the university educators is to provide assistance to overcome this attitude. In this direction, we cite Georg Pólya who said in 1961 (see section 14.9 of "Die Einstellung des Lehrers"): "Damit das Lehren des Lehrers zu dem Lernen des Schülers führt, muss irgendein Kontakt, irgendeine Verbindung zwischen beiden bestehen; er muss imstande sein, sich in die Lage des Schülers zu versetzen." It is, of course, too naive to assume that the relationship demanded by Pólya will be automatically implemented between the teacher students and the professors through the teaching of mathematics. Experiences and beliefs of the students as well as content knowledge, pedagogical content knowledge, and general pedagogical knowledge have to be taken into account by the university teach-

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ers and interconnected with the perspective profession of the students. The universities have to create the respective educational environment taking into account these various dimensions under the given constraints. As a consequence for the mathematics teacher students education, we are convinced that a thorough mathematical content knowledge should form the basis for the teaching of mathematics upon which and in synchronization with which the subsequent didactical, methodical, and pedagogical skills are then being built.

Based on this approach, we introduced at Humboldt-Universität zu Berlin courses that explicitly integrate content knowledge with pedagogical content knowledge. These courses consist of four-hour lectures in mathematics and one-hour lectures in the didactics of mathematics with two-hour and one-hour tutorials, respectively. In general, these courses are taught in cooperation of a mathematician and a didactician jointly planning the course as a whole. In these courses, the students reflect mathematical contents for the school from a higher viewpoint and are thus strongly motivated to construct their individual network of content, pedagogical content, and general pedagogical knowledge.

We will now illustrate two examples of such courses, namely the aims and contents of the courses "Stochastics and its didactics" as well as "Algebra/number theory and its didactics".

The course "Stochastics and its didactics"

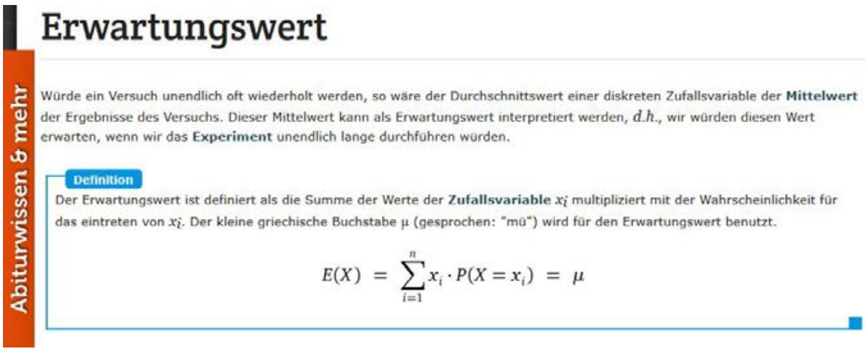
The mathematical part of the stochastics course is designed along the corresponding school curricula as well as the nationwide educational standards ("Bildungsstandards") for the stochastics education at schools. Thus, the main focus is on discrete probability spaces. This is in contrast to the corresponding course for the mathematics majors, which incorporates measure theory from the very beginning. Furthermore, the course encompasses distributions with densities with a main emphasis on the normal distribution. In particular, the teacher students experience the normal distribution as an approximation arising from the binomial distribution. Finally, the students are introduced to the concept of statistical testing. In summary, in this course, the teacher students should adopt fundamental competencies in the modelling of random phenomena by acquiring the basic notions as well as insights and conclusions characterizing stochastics.

The pedagogical content part of the stochastics course now builds upon these mathematical foundations. Along the keywords

- data and chance
- introduction to the notion of probability
- Laplace probability and basic principles of combinatorics
- multi-stage processes and path rules
- simulation and stochastic modelling
- conditional probability, independence and random variables
- binomial distribution and applications
- testing of hypotheses

teacher students are confronted with the prerequisites, the goals as well as with the planning and possible obstructions of instruction. The students are asked to develop sequences for teaching and to evaluate teaching material. Furthermore, they learn how to deal with possible reactions from high school students and to evaluate the students' performance. They are asked to analyse and interpret basic notations and to understand these in the context at large. As a concrete example, we mention the notion of the "expected value" of a discrete random variable. In the mathematical part of the course there is barely time to go beyond the definition and basic properties of this notion. However, in the didactical part of the course, we now expect the teacher students to reflect, for example, on the following questions:

a) Work out the definition of the "expected value $E(X)$" of a discrete random variable X with a finite range of values. Which information of the distribution of X contains $E(X)$, which information gets lost? Illustrate three different examples by using a graphical representation of the distribution of X.
b) Let $E(X) = 3$. Interpret this value by switching from the level of a mathematical model to the real world level. To which previous knowledge do you have to connect to?
c) Under http://matheguru.com/stochastik/166-erwartungswert.html one can find the following introduction of the concept of "expected value":



Erwartungswert

Abiturwissen & mehr

Würde ein Versuch unendlich oft wiederholt werden, so wäre der Durchschnittswert einer diskreten Zufallsvariable der **Mittelwert** der Ergebnisse des Versuchs. Dieser Mittelwert kann als Erwartungswert interpretiert werden, $d.h.$, wir würden diesen Wert erwarten, wenn wir das **Experiment** unendlich lange durchführen würden.

Definition

Der Erwartungswert ist definiert als die Summe der Werte der **Zufallsvariable** x_i multipliziert mit der Wahrscheinlichkeit für das eintreten von x_i. Der kleine griechische Buchstabe μ (gesprochen: "mü") wird für den Erwartungswert benutzt.

$$E(X) = \sum_{i=1}^{n} x_i \cdot P(X = x_i) = \mu$$

Assess the approach of this online mathematics learning platform to introduce the concept of "expected value".
d) Give a sketch of your ideas for the introduction to the concept of "expected value".
e) How is the concept of "expected value" connected to the education of using a critical reasoning? Where are the limits of the concept and its necessity to complement it by further concepts?

The course "Algebra/number theory and its didactics"

The mathematical part of the algebra/number theory course is again specifically designed for the education of teacher students. Here, the main focus consists in providing sound foundations for the arithmetical and algebraic contents to be taught at school from a higher point of view. Starting with the basics of elementary number theory and about algebraic structures, the teacher students are given a systematic treatment of the build-up of the number systems from the natural to the real and complex numbers. Alongside, the students learn about applications such as the RSA-cryptosystem or, more on the theoretical side, for example the transcendence of π. The course is designed according to the recommendations of the German Mathematical Society (DMV), the Society for the Didactics of Mathematics (GDM), and the Society for the Instruction of Mathematics and the Sciences at Schools (MNU) from 2008, in fact, it goes even beyond these recommendations.

In analogy to the stochastics course, the content knowledge part is complemented by a corresponding pedagogical content knowledge part, which provides an overview on the didactical concepts for the treatment of the following topics in primary school and on the lower secondary level:

- number systems – from the natural numbers to the reals
- algebra – variables, terms, equations, functions.

In particular, the course addresses the following questions:

- What are the pre-concepts of the learning pupils on the respective level (in particular, pre-school experiences and beliefs)?
- Which conceptions for the respective mathematical objects should the pupils develop? Which are important aspects in this respect?
- Which are standard obstacles in understanding? Which are frequent sources for errors? How can these be didactically handled?
- Which are canonical approaches, paradigmatic examples, and possibilities to connect these various aspects?
- Which of the overarching didactical concepts have proven to be meaningful?

The topic “construction of the number system” is prototypical for the conception of content-oriented learning. Wittmann (2014) describes it as follows: “The content-oriented building-up of learning is nothing but to bring to bear the established structures of mathematics on a long-term scale and across the various school levels. Thereby it has to be taken into account that mathematical structures are not only objects that hinder the learning process, they in fact provide at the same time the strongest support for learning since they stimulate the understanding.” However, mathematical structures deploy their supportive impact only, if the becoming teachers have well understood them and apply them in a goal-oriented way in class. Thus, the pedagogical content knowledge component of the course is devoted to use the knowledge that has been acquired in the mathematical part of the course in order to develop class concepts for a constructive mathematical learning process at school. Moreover, related methodological and pedagogical questions are also pursued in this context. The teacher students are thus requested to reflect in a goal-oriented way about the interrelationship of the content-oriented facets using the methodical and pedagogical dimensions of their knowledge.

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Oral examinations in first year analysis: between tradition and innovation

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We present the institutional and didactic rationales, as well as some results, from an intervention in a second semester course on real analysis at the University of Copenhagen, focused on improving the alignment between the course and its oral exam.

The problem: assessing theoretical knowledge in mathematics

Common introductory courses on calculus and linear algebra often focus on computational techniques, and are given in a variety of educational programs, from engineering to business. The average calculus text book is dominated by explanations of calculation methods and (especially) worked examples; they reflect a focus on technical knowledge, with standard techniques to be mastered and applied in a variety of tasks. Students get credit on the course if they demonstrate such technical knowledge (e.g. on finding extrema for a functions of two variables), usually during written tests with exercises that are "similar" to worked examples or exercises from the course. Such tests can be graded with a high level of consistency, and while the educational value of skills such as the one mentioned is open to debate, the assessment practice is well aligned with common teaching practices in such courses.

Students in pure mathematics may well take such "calculation oriented" courses at the beginning of their studies, but eventually they get more theoretical courses with titles like "abstract algebra", "topology" and "real analysis". In such courses, *theoretical structures*, built by definitions, theorems and proofs, form the core of the study material (textbooks, lecture notes etc.). There is little research on the common ways of assessing students' work with theoretical structures, on their effects, and on possible alternatives (cf. e.g. Grønbæk, Misfeldt and Winsløw, 2009). In some institutions, closed-book written exams are reported to be the norm, with questions like "State and prove X's theorem" (see e.g. Conradie and Frith, 2000, 225). In other institutions, such as Danish universities, the common form of assessment for theoretical knowledge in mathematics is the oral exam, based on questions of the same type (questions drawn at random from a list known in advance, with some preparation time between drawing and the actual examination). The two forms share major potential drawbacks, including a disproportionate effort by students to memorize proofs, and difficulties to provide and practice transparent criteria for grading. To counter these challenges, alternative ideas have been proposed for written and oral examinations in theoretical mathematics, focusing on students' proof comprehension (Cnop and Gransard, 1994; Conradie

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings.* Kassel, Germany: Universitätsbibliothek Kassel.
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and Frith, 2000; Mejia-Ramos et al., 2012) or even proof production (Grønbæk and Winsløw, 2007).

In this paper, we outline a number of specific and interrelated challenges which we encountered while investigating and developing the alignment between student work and examination practices in a first course on real analysis at the University of Copenhagen. We use basic terms and ideas from the anthropological theory of the didactic (those unfamiliar with the theory could consult Winsløw, Barquero, de Vleeschouwer, Hardy, 2014).

Institutional conditions and constraints

Our work is motivated in part by several years of experience with the common form of oral examination in post-calculus analysis that was outlined above. But there was also an institutional *condition* which fostered the project reported on here: at the level of the University (not even the Faculty, let alone Department) a wider development initiative to strengthen the quality of education by drawing on its basis in research (for a discussion of this general theme, see Madsen and Winsløw, 2007). At the Faculty of Science, four courses were selected to participate in this project, among them "Analysis 0", a second semester course on real analysis, which has had various operational problems including relatively high failure rates. Together with a colleague (who, like myself, is not involved in teaching the course), we observed five oral examinations at the conclusion of the course in 2014, which confirmed impressions from similar exams at this and other Danish universities: the examination format seems to work quite well for high performing students, who are able to deliver a presentation of a topic (such as "The Riemann integral") that is quite similar in autonomy and clarity to the lectures which are given in the course. However, the vast majority are not: they generally get stuck because they do not remember the "script" they have rehearsed, and whether they pass or fail depends then on the interaction with the examiner and in particular the degree to which they are able to pick up on his questions and "hints" as they struggle to formulate a proof (or even a basic definition). Doing a light search on the internet, I easily found a couple of pdf-documents giving a script for each of the exam questions, complete with advice on what to say and emphasize (anonymous, but most likely formulated by high-performing students). It thus appeared a likely hypothesis that many students relied on such "scripts" as much (if not more) than on their own reading of the official curriculum (text book, lectures etc.).

At the same time, during a later meeting with course teachers, we became aware of another problem the course has had for some years. While the lectures and exercise sessions are all well attended, the latter – conducted by teaching assistants – suffer from the fact that students do not prepare for the tutorials in the sense of even attempting to solve the exercises posed there. These exercises come from the text book and are theoretical, but according to the teaching assistants, students focus on rehearsing for the oral exam (on which all of their course grade is based), and they do not see the direct relevance of the exercises for this purpose; another obstacle lies, according to the teaching assistants, in the sheer difficulty of the exercises. In fact, several of the teaching assistants – who are some years further into the mathematics program – told us that they did not solve exercises when they, themselves, had the course as students. This means that the exercise sessions degenerate, to a large part, into lecture style presentations of solutions by the teaching assistants, with very minor contributions from the students and no visible impact on their final grade.

In a more advanced analysis course, we had previous, positive experiences with a variation of the oral examination, based on so-called thematic projects done as exercises (Grønbæk and Winsløw, 2007). However, other *constraints* – including the course responsible lecturer's concern about the high failure rates, and central role of the course in the first year curriculum for mathematics students – excluded such more drastic changes in examination format. We were left with the opportunity to redesign part of the exercises for the exercise sessions, to connect more directly to the exam questions. Moreover, it was decided that these new "F-exercises" ("F" being the first letter in the Danish word for "research") should provide experiences of research-like activities in relation to the exam question, and be declared "supplementary questions you may get at an exam on the corresponding question".

Relating calculus techniques and analysis theory

The contents of Analysis 0 can globally be characterized as supplying theoretical blocks corresponding to mathematical practices already worked on in the preceding calculus course and in secondary school (and going well beyond this as well). One of our strategies for designing F-exercises was thus to reinforce the links between these theoretical blocks and what could be assumed to be "old" knowledge of a more technical nature (cf. also Winsløw et al., 2014). Two assumptions motivate this choice: first, trying to mobilize old knowledge could be expected to provide a more realistic entrance level to the exercises, and thus reduce the previous experience of excessive difficulty. Secondly, theoretical blocks always draw their meaning and motivation from practical blocks.

We now outline a simple example illustrating these points. The second of the 12 exam questions is: "Taylor's formula in one variable". Corresponding F-exercises include:

F2.1. What is the Taylor series of order n for $f(x) = e^x$ at the point a? How can this be used to find a formula for e?

F2.3. Sketch a graph for a function g which satisfies (4.6) [*referring to an equation in the proof of a theorem in the notes, CW*] : $g(a) = g'(a) = \ldots = g^{(n\text{-}1)}(a) = g(b) = 0$, when: $a = 0$, $b = 1$, $n = 1$; and when $a = 0$, $b = 1$, $n = 2$.

In both cases, students' knowledge of familiar practices, such as computing derivatives and explaining their meaning in relation to a function graph, could be invested into the study of the meaning and proof of the theorem in question (the last exercise refers to the crucial step in the proof of Taylor's formula, in which Rolle's theorem is applied repeatedly).

Inducing research like activities among students

As already mentioned, the exercises also aimed to induce research-like activities in students' work. A basic one of these is to investigate simple questions, device hypothetical answers, and prove them; in the context of the question "Taylors formula", one F-exercise ask students to find out if the Taylor polynomial for $f + g$ and fg can be computed from the Taylor polynomial of f and g (with appropriate assumptions). Another research like activity is to generalize a known result. Associated with the exam question "The fundamental theorem of calculus", such an F-exercise was to try to extend that theorem to the case of functions $f: I \to \mathbb{R}^k$, where I is an interval and $k > 1$, as well as to piecewise continuous functions. Other examples relate more closely to the proofs given in the textbook, for instance to identify where an assumption is used or whether it is necessary, to elaborate on analogous cases omitted in proofs ("similarly it can be shown…"), devise alternative proofs in special cases, etc. Some of these exercises are actually close in nature to previous proposals mentioned in the introduction, although we had to consider the specific constraints related to the oral exam.

Outline of observed results

As part of an MSc-thesis work by Gravesen (2015), all exercise sessions in one class were videotaped and transcribed, and the oral exams of 32 (out of 300) students were observed; moreover, a focus group of students was followed more closely, for instance through the collection of homework during the course, and interviews after the exam. Moreover, a questionnaire on the F-exercises was filled by some students, and we have some evidence from exam statistics. Here, we can only provide some overall tendencies of the results.

Students' work before and during exercise sessions.

The problem of students' non-preparation for the exercise sessions was partly solved. Based on observation and questionnaires, we estimate that an average of about 40% of the students had worked on any given F-exercise before the sessions, in the sense that they brought notes for that exercise with at least a partial solution. It should also be noted that less technical parts of the F-exercises, like the second question of F2.1 cited above, frequently get oral contributions even from non-prepared students, based on what is presented in class. In the example, once the Taylor series has been developed on the blackboard, it is not so difficult to get the idea of choosing $x = 1$ and $a = 0$ to produce an approximation of e and to discuss how the error can be estimated using Taylor's formula. During the exercise sessions, time pressure led instructors to do almost all blackboard presentation of solutions, with only oral contributions requested from participants; the students who had prepared solutions at home were therefore not able to present more than oral indications of them. This emphasizes a more general challenge with the role of TA: frequently, the students and the TA both expect that it is the job of the TA to present solutions. There is a clear need for deliberate orientation (if not training) of the TAs if this should change significantly.

Students' point of view on F-exercises

According to interviews during the course and a survey at the end of the course, students are somewhat divided on whether the F-exercises represent a positive addition to the course. Naturally, they cannot compare with exercises given in previous editions of the course; so they sometimes refer to what older students, including teaching assistants, may have told them (which could be quite imprecise).

Most of the negative reactions seem to arise from the impression that the new exercises are "more difficult" or "weird". The difficulty is in fact experienced by many: only 16% agree that "I could solve almost all the F-exercises before the sessions" while 41% agree that "I normally try to solve all F-exercises before the sessions". We should recall, however, that in previous years, almost no students solved exercises before sessions, and there is good reason to believe a part of the reason was the experienced difficulty of the assigned exercises. There are also many (39%) who find the workload in the course excessive, and among those, we find relatively many students who see F-exercises as an unwelcome ”extra burden” (”additional syllabus”) rather than as a support measure. There is no doubt that the indication of the F-exercises as a “supplementary question” for the exam plays a role here – while it may increase the motivation to actually work on them, it is a double-edged sword.

On the positive side, students like that it is often possible to solve F-exercises ”partly”. At interviews, students also indicated that they were ”suitable for discussion” both with fellow students and in class, even in cases where the students had not tried or been able to solve the exercises beforehand. While preparing for the exam, the students who were interviewed all worked with F-exercises in relation to each question, ”reflecting on which to use or prepare for”. At this point, they are particularly interested in those F-exercises which focus on dissecting a proof which appears in one of the exam questions, such as F2.3 given above.

In general, students are divided on the utility of F-exercises in the course, in particular on the extent to which they see them as a research-like experience.

Students work and results at the exam

At the exam, the inclusion of F-exercises seemed to be most helpful for students who passed. Examiners included F-exercises more often in the examination of students who ended up with passing grades. Some exercises, like F2.1 above, were frequently used at the exam by students and examiners.

Students who end up with low or failure grades do not get to be asked F-exercise related questions. They often display a highly retrocognitive relationship to knowledge (they actually use the verb ”remember” frequently). As several of these students get stuck with the statement of even basic definitions, more F-exercises focusing on the meaning and motivation of definitions might be helpful.

By contrast, well-performing students often (implicitly or explicitly) use observations or examples which are related to F-exercises, without being prompted to do so by the examiners. This and other observation gives reason to believe that the F-exercises could be a partial explanation for the (compared to earlier years) significantly higher average of grades received by students who passed, while we find it less likely that they would be an important cause for the slightly higher passing rates (cf. Fig. 1). But it is impossible to measure the causal power of the F-exercises in relation to other changed factors, for instance that the lecturer published pen casts of selected proofs.

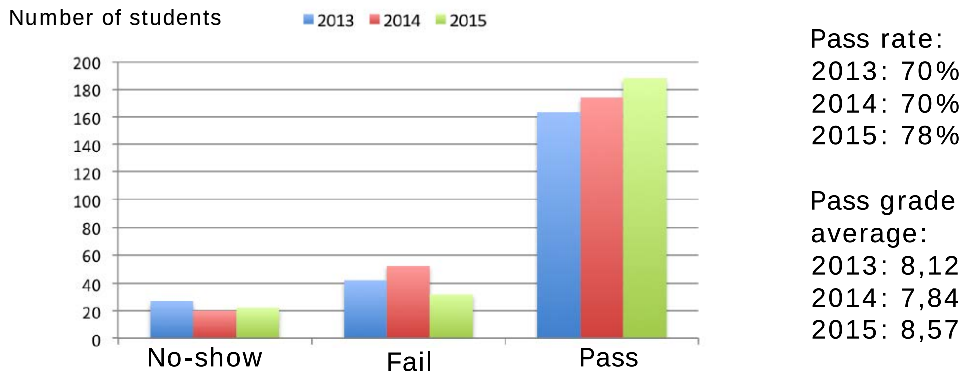

Figure 1. Exam results in Analysis 0, 2013-2015.

Examiners make use of certain F-exercises frequently while others are never evoked. Most popular are F-exercises filling some ”gap” in the material (like showing how to generalize the proof of Gauss' theorem from rectangles to more general regions).

Conclusions

In this paper, we have posed and outlined a first investigation of the following question:

1. How to assess theoretical knowledge (e.g. related to continuous functions: definitions, results, proofs or at least ”proof ideas”) in ways that do not simply lead to a pointless attempt at memorization for many students?
2. And, if the “traditional reproduction format” is given and cannot be changed, what can be done to avoid students getting trapped in the ”memorization trap” (working little during the course, and failing on last-days rehearsal to the exam)? Could new forms of XQ, with more explicit link between familiar practice and theory blocks, be viable?

Our first results with the last-mentioned strategy were reasonably positive, while we have pointed out the need for teachers of the exercise sessions (in our case, student assistants) to be much better prepared for leading and motivating the students' presentation of their work. We also believe that in working with the above questions, a crucial strategy – or even a major research programme – is to work on the *design of student assignments*, with the double aim of *strengthening students' command of theoretical knowledge* (in formal settings) and providing a better basis for the assessment of students' knowledge than the simple “official” text (giving definitions, proofs and theorems).

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8. THEORIES AND RESEARCH METHODS

Theoretical approaches of institutional transitions: the affordances of the Anthropological Theory of Didactics

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In this contribution, I discuss the affordances of the Anthropological Theory of Didactics (ATD) for approaching transition issues in mathematics education: transition between secondary education and university, or along mathematics university courses. For developing this discussion, after a brief introduction to ATD, I especially use three doctoral theses I have supervised in this area, those of Frederic Praslon on the transition regarding the concept of derivative and its environment between high school and university in France, of Analia Bergé on the evolution of the relationship to the field of real numbers and completeness along university mathematics courses in Argentina, of Ridha Najar on the transition regarding functions between high school and selective university courses in Tunisia.

ATD and transition issues

Several reasons make a priori promising the use of the Anthropological Theory of Didactics (ATD) to approach issues of institutional transition and diversity, among with the three following reasons:

- From the first presentations of ATD by Chevallard (Chevallard, 1992), emphasis has been put on the institutional nature of relationships to mathematics knowledge, and their subsequent relativity. In any institution, identifying a particular object as an object of knowledge, let us say for instance complex numbers or differential equations, knowing this object means something specific. For any subject of this institution, knowing this object means showing a personal relationship close enough to this institutional relationship, which is generally partially differentiated according to the different positions that subjects may occupy in the institution (for instance, student and teacher position).
- Through the modeling of human practices in terms of *praxeologies* (Chevallard, 2002), (Chevallard & Sensevy 2014), ATD provides operational tools to investigate what the differences of institutional relationships exactly consist of, at the level of both praxis (genres and types of tasks and techniques) and logos (technological and theoretical discourse) and of their relationship, and to question how these differences are treated in transition processes.
- ATD also pays particular attention to the diversity of conditions and constraints which, at very different levels, shape what can be taught and learnt in a given institution, and how. The idea of *hierarchy of levels of didactic codetermination* operationalizes this sensitivity (Chevallard, 2002).

The use of ATD thus a priori helps us to question the priority often given to cognitive approaches and interpretations when studying the difficulties of institutional transitions in mathematics education, and instead to envisage students' difficulties as the sign of unno-

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ticed or underestimated institutional breakdowns. Doing so often opens the way to original and effective educational strategies. A pioneering work in that area, the doctoral thesis regarding the transition between vocational and general education by Grugeon (1995), already showed this twenty years ago. Since that time, various pieces of research confirmed the potential of ATD for the study of institutional transitions. In this contribution, after briefly presenting the concept of praxeology and the hierarchy of levels of codetermination, I will especially use the reflective analysis of three doctoral theses that I have supervised regarding teaching and learning processes at university or at the transition between high school and university to discuss ATD affordances.

Praxeologies and hierarchy of levels of didactic codetermination

Praxeologies

In ATD, as pointed out above, mathematical practices, as any kind of human practice, are modeled in terms of a four-component structure called praxeology. The first component is a *type of task*, for instance, f being a real valued function, and a, b, c real numbers, to prove that the equation $f(x) = c$ has a unique solution on the interval $I = [a, b]$. The second component is a *technique*, a way of performing the task. For the type of task just mentioned, a technique taught in high school is the following: To show that f is continuous and strictly monotonic on I and that c belongs to $[f(a), f(b)]$ (resp. to $[f(b), f(a)]$). The third component is the *technology*, a discourse explaining and justifying the technique. In our case, for instance, the Intermediate Value Theorem is part of this technological discourse. The fourth component is the *theory* which justifies the technology itself, here the theory of functions of one real variables or a part of it. This structure describes the most elementary form of mathematical praxeology also called *punctual praxeology*, but these coalesce into *local praxeologies* sharing a common technological discourse, and the local praxeologies themselves into *regional praxeologies* sharing some common theoretical ground, to build complex *mathematical organizations*.

Hierarchy of levels of didactic codetermination

As mentioned above, the hierarchy of levels of didactic codetermination helps us to better understand the complex system of conditions and constraints which condition the ecology of mathematical and didactical praxeologies. Nine levels are distinguished, situated above and below the discipline level (in our case mathematics). Any mathematical object whose teaching and learning is at stake in a given institution situates within one *domain* of mathematics, which itself is divided into several *sectors* made up of different *themes* or *topics*, that can be separated into different *subjects*. In the French high school curriculum for instance, the topic of the variation of exponential functions is part of the sector of functions, which is part of he domain of elementary analysis, and this influences the corresponding praxeologies. However, these praxeologies are also shaped by a diversity of conditions and constraints, beyond the inscription of the topic in these particular sector and domain. Hence, in the hierarchy, the existence of levels above the discipline level, the respective levels of *pedagogy*, *school*, *society* and *civilization*. In (Alves et al. 2010), comparing the respective expectations regarding the learning of sequences and functions at the end of high school between Brasil and France, we have shown the potential offered by this conceptual tool to describe the respective mathematical organizations and to understand the "raisons d'être" of the important differences observed.

Frederic Praslon's thesis: the concept of derivative in the transition between high school and university

Praslon's thesis (2000) was the first thesis using ATD to study the secondary-university transition. Praslon approached this transition by focusing on the concept of derivative and its environment, being it the core concept in high school elementary analysis. As had been the case for Grugeon, he used ATD as a macro-theoretical framework to question the vision of the secondary-university transition prevailing at that time, which naturally led him to study and compare the praxeologies involving the concept of derivative in the two last years of scientific high school and in the first university year. As was also the case in Grugeon's thesis, he combined this macro-theoretical framework with constructs such as the distinction between the tool and object dimensions of mathematical objects[1] due to Douady (1986), the idea of semiotic register due to Duval (1995), and the ideas of procept and mathematical world of Tall (2004). These helped him to build on already established knowledge regarding the teaching and learning of Calculus and Analysis. The praxeological analysis he developed followed a standard methodology based on the quantitative and qualitative analysis of a diversity of curricular resources (syllabuses, textbooks, student worksheets, assessment tasks, etc.).

This praxeological analysis first showed that a substantial universe around the notion of derivative already developed in high school, but that a dramatic enlargement of the landscape was taking place in the first six months at university. Praslon visualized this phenomenon through the use of insightful concept maps. He also showed that, contrary to what was often said at that time, the secondary-university transition in this area was no longer, at least in France, a transition from the proceptual world to the formal world, or from intuitive to rigorous approaches. The difficulties of the transition resulted more from an accumulation of "micro-breaches", less visible and not appropriately taken in charge by the institution.

These breaches affected particularly the balance between the tool and object dimensions of the derivative, between the study of particular objects and objects defined by general conditions, and between algorithmic techniques and techniques having more the status of general methods to be adapted to each particular case. Results were more systematically proved, and proofs were no longer "the cherry on the cake" but took the status of mathematical methods. Shifts also resulted from the increased autonomy given to students in the choice of appropriate mathematical settings and semiotic registers, and more globally in the development of the solving process. He also observed an impressive diversification of tasks and techniques, and acceleration of the didactical time.

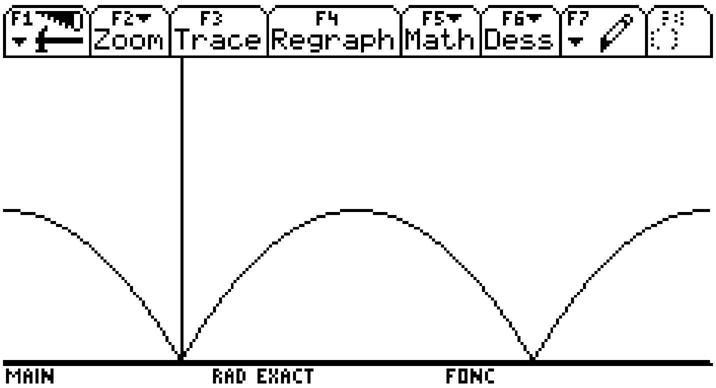
From an institutional culture organized around the mastery of a restricted number of punctual praxeologies that could become reasonably familiar, students moved to a culture in

[1] A mathematical entity may be considered as a tool used to solve different types of tasks, and also as an object which is part of a structured set of mathematical objects.

which praxeological diversity was the norm and the acceleration of didactical time made routinization much more difficult. The conjunction of these breaches created a substantial gap but university teachers were not aware of it in their great majority and tended thus to under-estimate the cognitive charge induced for their students. I would like to point out that in this pioneering research results are expressed by using a language not strongly tied to ATD. The difference is evident when one compares with the research developed by Bosch, Fonseca and Gascón (2004) on close issues. In my opinion, this difference illustrates the fact that in this thesis, used as a macro-theoretical framework, ATD guides the questioning but still moderately instrumentalizes research praxeologies (Artigue, Bosch & Gascón 2011).

For making university teachers and students sensitive to these changes, Praslon designed a set of tasks in the gap between the two institutional cultures, and revealing main facets of the differences between these. For instance, none of the tasks required the use of $\varepsilon - \delta$ formalizations. They were proposed to the students before their entrance to university or at the beginning of the academic year, and were used as a base of discussion with university teachers. We give below an example of such a task, explaining why it situated in the gap between the two institutional cultures at that time.

Let us consider the periodic function f with period 1 defined by $f(x) = x.(1-x)$ on $[0,1[$.



The first question (Q1) asks: Is this function continuous? Differentiable?
The second question (Q2) formally introduces the notion of symmetric derivative at a point x for a function f as the $lim_{h\to 0} \frac{f(x+h)-f(x-h)}{2h}$. Then students are asked to compute the derivatives and symmetric derivatives of f, if they exist, at points ½, ¼ and 0, and to compare these.
The third question (Q3) asks them to say if the following three conjectures are true or false and to justify their answers:

- Every even function defined on IR has a symmetric derivative at 0.
- Every even function defined on IR has a derivative at 0.
- If a function defined on IR has a derivative at a, it has also a symmetric derivative, and the two are equal.

At the time Praslon was working on this thesis, a French student entering the University after a scientific baccalauréat (Bac S) a priori has been taught the mathematics required for solving this task. Nevertheless this task was not part of the high school culture in France. For instance, f is defined by pieces and the students have to understand that the given expression can only be used on [0,1[. They have certainly already met such functions but these have remained marginal objects. Question Q1 is not a new question, all the more as the students are given a graphical representation for supporting their reasoning, but this is not at all a routine question. In question Q2, a new notion is introduced through a formal definition and the students are asked to make sense of this definition. Question Q3 proposing three general conjectures is also rather unusual even if answers to previous questions give important hints.

Ridha Najar's thesis: the concept of function in the transition between high school and university

The thesis by Najar (2010) also regards the transition between high school and university, with students for whom a priori the transition should be easier, as they were top-level students in high school, are confident in their capabilities and very committed. They study in the selective program of "Classes Préparatoires Scientifiques" (CPS) in Tunisia. Once again, ATD is the main theoretical framework, and its constructs are used to elucidate the change in the relationship with the notion of function. What Praslon's thesis could not show and is made clear by Najar's research is that a main source of discontinuity in institutional relationships regarding functions in the secondary-university transition is the move from praxeologies mobilizing functions of one real variable for solving Calculus tasks to praxeologies involving functions conceived as set theoretical objects or homomorphisms between algebraic structures. The detailed analysis carried out shows the huge distance separating these praxeologies, the techniques, the modes of reasoning and of using semiotic resources, they respectively engage. The type of task "Proving that a function is a bijective mapping" well illustrates this difference, when one compares the technique favored in high school, based on the study of variations of the function and the technological property that a continuous function f strictly monotonous on an interval I of IR is a bijective mapping from the interval I onto the interval $f(I)$, and those used in set theory or abstract algebra, coming back to the definition or using specific characteristics of homorphisms in abstract or linear algebra.

The example below is a typical example of task proposed to students in the first worksheet on set theory and functions. It illustrates this difference.

E, F, G and H are sets and H has two elements at least, f is an element of $A(F,G)$, the set of applications from F to G; prove the following equivalences:

$$f - \text{surjective} \Leftrightarrow [\forall g,h \in A(G,H), (g \circ f = h \circ f \Rightarrow g = h)]$$

$$f - \text{injective} \Leftrightarrow [\forall g,h \in A(E,F), (f \circ g = f \circ h \Rightarrow g = h)]$$

In it, not only new techniques must be used based on the definition of injective and surjective mappings, but to this adds the complexity of the statements to be proved. These are two equivalences whose second term is itself an implication universally quantified, which highly increases the difficulty of the technical work to be developed to implement these techniques successfully.

As shown by Najar, contrary to France, in Tunisia even if the main habitat for functions in high school is Calculus, set theoretical perspectives are already present in the teaching of geometrical transformations. However associated praxeologies consist just of a few isolated and rigid punctual praxeologies and their technological-theoretical block is exclusively in the teachers' topos, in other word under the teachers' responsibility. Comparing with CPS makes the praxeological breakdown clear. In CPS, crucial importance is given to the technological-theoretical block, and a diversity of techniques are suddenly associated to the same type of tasks. This is in line with the analysis of institutional transition provided in (Bosch, Fonseca & Gascón 2004). Najar shows that curricular resources and teaching practices pay little attention to these changes, to the associate development of set theoretical language and mathematical symbolism, and more globally to what Castela (2008) calls the practical component of the technology. As was the case in Praslon's thesis, the results of this praxeological analysis are reinvested in a didactical engineering, in that case very much constrained by the specific context of CPS classes. Emphasis is put in it on the blind points that the institutional study has revealed.

As was the case in Praslon's thesis, Najar combines ATD with other theoretical constructs, for instance those developed by Tall already mentioned, and those developed by Robert (1998) for the analysis of tasks distinguishing between different levels of use of mathematical knowledge and different types of adaptation required by the solving of these. However, a careful look at Najar's thesis shows that he makes a more advanced use of the construct of praxeology than Praslon, including the distinction between punctual, local and regional mathematical praxeologies, the reference to the idea of completeness of local praxeologies introduced in (Bosch, Fonseca & Gascón 2004) and the use of associated criteria. This more advanced use is visible both in the organization of the research work, and the expression of its results.

Analia Bergé's thesis: conceptualization of completeness along university teaching

Bergé's thesis (2004, 2008) studies students' evolution in the conceptualization of the field of real numbers and completeness along university mathematics courses at the University of Buenos Aires. ATD is used in the institutional part of this research work, also supported by a deep historic-epistemological study leading to a Mathematical Panorama complemented then by a Cognitive Panorama structured around six axes of development with a common origin (an initial state where completeness is considered an evident property of the Real number field). This cognitive panorama is used as an analytic tool for the praxeological analysis and that of students' questionnaires and interviews.

The praxeological analysis shows that in the first university course dealing with real numbers, completeness remains in a pre-construction state; it lives encapsulated in powerful theorems such as the Intermediate Value Theorem, and is implicit in the use of graphical representations. A radical change occurs when students enter the course "Analysis I" where graphical representations are no longer allowed as support for argumentation. However the praxeological analysis shows that the tasks involving completeness remain essentially the same with, as a consequence, the fact that students interpret this change merely as a change in the didactic contract. In "Complements of Analysis II", completeness becomes an official objet of study through the notions of supremum and infimum, and in "Advanced Calculus" the idea of completeness is further deepened in the context of metric spaces. As pointed out by Bergé, these different courses function as isolated institutions, and connections between the perspectives they offer on completeness are left to the students' private work. Her praxeological analysis also reveals that two axes of development are not explicitly addressed in the institutional offer, and also let to the students' private work. Questionnaires and interviews with students having successfully passed these different courses allow Bergé to describe their personal relationship and its evolution along university courses. I will not enter into the results of this analysis but it clearly shows that these courses do not allow students, in their great majority, to understand the role of completeness for developing mathematical analysis.

Final comments

In these three examples, ATD is used as the main theoretical framework. As they make clear, this theoretical choice shapes the way research questions are articulated and the methodologies used to work them out. Difficulties in transition processes are mainly approached in terms of discontinuities in institutional relationships to mathematical knowledge. In the search for such discontinuities, the analysis of mathematical praxeologies is given a fundamental role, with equal importance given to the praxis and logos dimensions of praxeologies, particular attention paid to the way these two dimensions are related, and to the respective topos of students and teachers. The three examples show that such an approach can be insightful, attract the attention on blind points of the transition process, and inspire effective didactic strategies. These examples also show that the use of ATD as a global theoretical framework can be productively combined with the use of theoretical constructs of a different origin at a more local level. These theoretical combinations visibly help researchers to take into account the affordances of educational research on the specific mathematical domains at stake and the results of their epistemological analyses. Carefully managed, they contribute to the operationalization of research practices piloted by ATD.

They three theses show an evolution in the use of ATD as far as the theory itself and its use for addressing transition issues progresses. However, in these theses ATD is only used as a descriptive theory. The more recent developments of ATD oriented towards design, in term of Activity of study and research, and Paths of study and research (PSR), which play an essential role in the theses by Barquero (2009) and Oliveira (2015) for instance, are not invested in the didactic engineering work developed by Praslon and Najar. In the first case, this is normal as the design dimension of ATD did not exist at that time, and for Najar the strong institutional constraints of Classes préparatoires certainly made difficult if not impossible the organization of a PSR, a form of didactic intervention so distant from the traditional ones. These theses also show researchers that situate themselves in a user relationship to ATD, and make the choice of combining ATD with constructs external to it to fulfil their specific needs, instead of envisaging contributing to the development of the theory itself. One can understand that doctorate students do not necessarily feel themselves legitimate to envisage such an authorship position with respect to well established theories as ATD is, but I would like to mention that this limitation can be overcome, as shown by the doctoral thesis of Romo Vazquez that I supervised together with Castela (Romo Vazquez 2009). What certainly made the move possible in that case was the position of Castela herself with respect to ATD.

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Study and research paths in university mathematics teaching and in teacher education: open issues at the edge between epistemology and didactics

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During this past decade, many investigations conducted in the Anthropological Theory of the Didactic have focused on the design and analysis of a new teaching proposal—study and research paths, SRP—linking inquiry-based activities with the study of contents. SRPs have also been recently used in teacher education as a way to provide teachers with new epistemological models of the mathematical domains and topics that are to be taught. In this context, SRPs appear as both teaching devices and analytic tools to question and reconstruct curricular mathematical contents. The links between SRPs and the construction of alternative epistemological models of the contents to be taught open new research questions at the edge between epistemology and didactics.

In the research tradition of Didactics of Mathematics promoted in the 1980s by the Theory of Didactic Situations (Brousseau 1997) and followed by the Anthropological Theory of the Didactic (ATD) (Chevallard 1992, 2014), the epistemological dimension has always been at the core of the study of teaching and learning phenomena. By this we mean not only the consideration of the knowledge that is taught and learnt, but a real questioning of it, to avoid assuming the viewpoint provided by the institutions where the production, development and dissemination of this knowledge take place. We here present three investigations in ATD at university level that show different aspects of the researchers' detachment from the institutions where teaching and learning processes take place and, more especially, from the "scholar" one, responsible of the production and conservation of knowledge.

From "Workshops of practice" to "study and research paths"

In the 1990s, the first investigations in university mathematics education carried out by our group of research focused on the implementation of a new teaching device called "Workshops of practice" (WoP) in a Mathematics degree, aiming at overcoming the two-fold classical organization of university teaching in "lectures" and "problem sessions" (Bosch & Gascón 1993). A WoP consists in 3 or 4 weekly sessions of 3 hours proposed to deeply study a single type of problems taken from the problem sessions, containing a set of several cases all similar but presenting their own specificity. Its main goal is to give visibility to the "technical work", which is crucial to mathematics creativity. Students explore by themselves a set of problems they initially know how to solve but that now and then require a slight or important variation of the technique. This kind of work produce new theoretical needs related to the scope of the technique and the limits of the type of problems approached. In a sense, the WoP connects lectures to problem sessions in a reverse way: instead of "applying" new

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knowledge to new types of problems, it is the deep exploration of a single type of problems that opens new questions and new theoretical needs.

WoPs were experimented during several academic years in different mathematical subjects, from linear algebra or calculus to complex analysis, and their design required a real deconstruction and reconstruction of the traditional organisations of contents based on the logic of concept construction. One has to identify the main types of problems that constitute the core of each subject and choose a set of problems of a given type that can show the functionality of the knowledge introduced. For instance, the Jordan decomposition of matrices was "practiced" through the general problem of finding the n-th power of a set of matrices; the power series development of functions through the resolution of ordinary differential equations; limits through the ranking of functions according to their convergence speed; etc.

WoPs produced many disruptions in the normal university didactic contract. However, they preserve the main features of the paradigm of "Visiting works" (Chevallard 2012) that prevails in university teaching: students are introduced to a well-chosen sample of mathematical "monuments" they are supposed to learn, the rationale of which remains implicit: why calculating the power of a matrix, why ranking functions or finding isomorphisms between finite groups? Teachers pose problems, students solve them.

SRPs "covering" first courses of Mathematics as service subject

From 2005 on, the developments of the ATD head towards the new paradigm of "Questioning the world" by studying the conditions for and effects of introducing another teaching device called "Study and research path" (SRP) in different school settings, and especially at university. In this case, the aim was to reverse the priority given in the traditional paradigm of "Visiting works" based, at university, in the presentation of new contents (lectures) before their practice (problems solving sessions). SRP propose a completely different and complementary activity focused on a long inquiry study of a problematic question that is to be run in parallel with the ordinary lessons and motivate the need of new knowledge. A SRP starts with the consideration of a problematic *generating question* and follows a complex sequence of derived questions and provisional partial answers, including both the construction of appropriate experimental *milieus* (in the sense of non-intentional systems), and the access to previously available knowledge works. The paradigm of "Visiting works" can be subsumed into the general frame of SRP in its *study* dimension: during the inquiry process, some piece of knowledge may seem useful (or at least usable), and a specific lecture can be the best way to make it available in order to let the *research* progress (Winsløw et al, 2013).

The implementation of SRP with first-year university students has been tested in two institutional settings and under different conditions. The first one (Barquero et al 2008, 2011, 2013) was experimented during five academic years (from 2005/06 to 2009/10) in the one-year 'Mathematical Foundations of Engineering' course of a technical engineering degree (a 3-year programme). The SRP consisted in a workshop of 2-hour weekly sessions and was ran in parallel with the usual lectures (three 1-hour sessions per week) and problem sessions (1-hour session). The generating question (Q_0) consisted in predicting a population dynamics given its size over some previous periods of time. It also supposed to wonder about the assumptions on the population, its growth and its surroundings that should be made. In all its implementation, the workshop focused on this initial problematic question Q_0 to which students had to provide a complete answer during the entire academic year. Q_0 was presented using different populations: pheasant, fish, and yeast populations. To provide some answers to Q_0 and to the sequence of derived questions that followed it, the construction of different mathematical models was required. Depending if time was considered as a *discrete* or *continuous* variable, different types of models were developed. Due to official curricula conveniences, the lecturer assured that during the inquiry process questions about discrete models with mixed and independent generations, as well as continuous models were tackled, which were respectively approached during each of the three terms of the course. At the end, this sequence of modelling activities required the activation of most of the contents of the course: one-variable calculus, linear algebra, ordinary differential equations and their systems. Moreover, this activation took place in a functional way, that is, to provide answers to the questions raised during the inquiry process.

Following the same design, other SRPs are being implemented since 2006 in a Mathematics course of a first year degree in Business Administration and Management (Serrano et al 2011; Serrano 2013). The course is divided in three terms: one-variable calculus, two-variable calculus and linear algebra, and the contents are organised following a modelling perspective: families of functions, rates of change, optimisation, etc. In this case, the SRP is carried out during a 90 minutes weekly workshop and is more "organically" connected to the two other 90 minutes lecture-and-problem-solving sessions: it is always the same teacher who is responsible of all the sessions and the problems approached outside the workshop are more similar to the questions raised and studied in it. In a sense, the SRP consists in taking "more seriously" and in a more professional context one of the main problems that are studied in the course. The generating questions approached vary from one year to the other (sales forecasts, urban bike business, social networks, credits and loans, etc.). They share certain invariants in order to avoid going far away from the course contents: search for empirical data and forecast using functional modelling (first term); several variable formulae (second term); evolution of Markov processes or linear multidimensional phenomena (third term).

Study and research paths for teacher education (SRP-TE)

SRPs have also been recently used in pre-service and in-service teaching education as a way to provide teachers with alternative models of school mathematical activities (Barquero et al 2015). In this context, a SRP-TE appears as both a possible teaching device to be implemented under special conditions and an epistemological tool to question the established knowledge and reconstruct it as a dynamical process of questions and answers. We can thus consider them at the core of the so-called "mathematical content for teaching". The starting point of a SRP-TE is an open question related to the teaching of a given piece of knowledge. The second stage consists in presenting a SRP related to the initial teaching question approached: for instance a SRP on savings plans related to the open question "How to teach proportionality?". After carrying out the SRP as students, teacher-students are asked to describe the mathematical process followed and the specific didactic conditions that made it possible or hinder its development. This epistemological and didactic analysis is followed, when possible, by the adaptation of the SRP to real school conditions and a posteriori analysis of its implementation. What appeared to us as especially important in the few experimentations of SRP-TE carried out is the possibility for the teachers to use the description of SRP in terms of sequences of questions and answers to provide and alternative description of a whole block of mathematical contents and, furthermore, to relate domains that usually appear as disconnected in school mathematics (statistics and functions, for instance).

Open issues

When trying to adapt SRP to school curricula, the knowledge to be taught, which is usually conceived as a well-structured organisation of concepts and problems, needs to be completely reformulated in terms of an arborescence of possible questions and provisional answers. This tree-maps of questions and answers constitute the skeleton of potential SRPs to be carried out with students. We are thus at the core of the epistemological analysis of deconstruction and reconstruction of mathematical knowledge, based on a clear detachment from the dominant (scholar and school) conceptions of what mathematics is and how it should be organised. However, what appears as a powerful tool for the analysis becomes at the same time one of the main weaknesses of SRP as teaching and learning activities. The experimentations carried out, mainly by researchers acting as teachers or teacher educators, show a shortfall of available epistemological resources, not only to design and manage SRP (How to find the generating questions? How to motivate their study?) but even to *talk* about the inquiry work that is done and the results produced. This work cannot always be described in the traditional terms used in lectures and problem solving sessions: the specific derived questions and partial answers obtained during the process need to be named, classified, structured, and the official mathematical resources are not always appropriate. Teachers, together with students, need to invent their own terminology and propose a specific organisation of the different conceptual and technical tools that are used, as well as of the results obtained, which cannot always be located in the universe of shared scientific works. This is one among the many problematic issues that this new approach opens, in a research field where epistemological and didactic questions become more and more difficult to extricate.

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When praxeologies move from an institution to another: an epistemological approach to boundary crossing

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The issue of vocational mathematics education is commonly approached through a vigotskian lens focusing on the individual development within different socio-cultural contexts: how does one student tackle the experience of crossing boundaries between mathematics, engineering sciences, vocational training? In my presentation, I take the opposite yet complementary point of view, that of the anthropological theory of the didactic, which emphasises the social and institutional dimensions. Drawing on the notion of praxeology as a model for socially acknowledged cognitive resources of institutions, I develop this model to address an epistemological issue: how are mathematical praxeologies transformed when crossing institutional boundaries? Examples refer to mathematics and automatic control.

Introduction: The Anthropological Theory of the Didactic

The anthropological theory of the didactic (hereafter ATD) is at the same time a theory and the most prominent dimension of a research program in mathematics education. This program has been initiated by Yves Chevallard (1985) with the study of didactic transposition processes, the anthropological perspective being introduced in 1992 (Chevallard 1992). A socio-cultural conception of humans underpins the ATD, with a focus on institutions (stable social organisations) as absolute precondition for humanity's development and social activities. Institutions foster collective processes for facing and solving human problems. They favour the dissemination of innovations and more widely provide the necessary resources (material and cultural) for activities to take place. Conversely each institution constrains the different types of activities that it expects people to carry out in the social environment it builds. An individual has to satisfy the institutional expectations, to a certain extent at least, depending on the institution; that is why he is considered as an institutional subject (from Latin sub-jectus: literally thrown under) when acting within this institution. Hence, the ATD considers that human activities are institutionally situated and, consequently, so is knowledge about these activities. When a fragment of social knowledge, produced within a given institution I, moves to another one I_u in order to be used, the ATD's epistemological hypothesis states that such boundary crossing most likely results in some transformations of knowledge, called transpositive effects. Any didactical institution I_d that intends to train students to meet I_u's demands should be aware of these changes from I to I_u ; otherwise they will leave the full responsibility of knowledge adaptation up to the students. Moreover, let us recall that the specific nature of activities within I_d, under specific constraints, results in other transpositive effects, the so-called didactic transposition.

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Praxeology

The key notion of praxeology is the basic unit proposed by ATD to analyse the institutionally acknowledged capitals of practices and knowledge (see Chevallard 2006, p.23). A praxeology entails two interrelated components, *praxis* and *logos*. The practical block (or know-how) associates a type of tasks T and a technique τ. τ is a "way of doing" which is endowed with certain efficiency for a certain subfield within the set of T tasks. The *logos* block contains two levels: the technology of the technique (θ) gathers the whole rational knowledge referring to the technique; the theory (Θ) is a second level of more general knowledge supporting the technological discourse.

To exemplify the praxeological model and give an idea of its potential to analyse the transpositive effects of boundary crossing, I will consider a mathematical type of tasks encountered in strictly mathematical contexts as well as engineering sciences: *Breaking up a rational function into partial fractions*.

Mathematics praxeologies for breaking up a rational fraction into partial fractions

Let me emphasise that by mathematics praxeology I mean a praxeology that is acknowledged in the current institution of research in mathematics. Hence we will refer in this part to the mathematics' norms for proof. The following technological elements are derived from the analysis of a calculus online textbook[1].

Description of the technique in the general case: (1) Make the denominator monic (leading coefficient 1), and use the Euclidean algorithm to reduce to a problem where the degree of the numerator r is less than the degree of the denominator d. (2) Factorise the denominator as a product of powers of distinct monic irreducible polynomials. (3) Write the fraction as a sum of partial fractions of the form R/Q^k, where Q is one of the irreducible factors, k is at most equal to the multiplicity of Q in d and the degree of R is less than the degree of Q. (4) The coefficients of every R need to be determined. One way of doing this is to take a common denominator, multiply out, equate coefficients and solve the resultant system of equations.

Example: We want to express $\frac{3x+1}{(x-1)^2(x+2)}$ as the sum of its partial fractions $\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{x+2}$.

$$\frac{3x+1}{(x-1)^2(x+2)} = \frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{x+2} \Leftrightarrow 3x+1 = (A+C)x^2 + (A+B-2C)x - 2A + 2B + C$$

$$\Leftrightarrow \mathrm{A}+\mathrm{C}=0,\ \mathrm{A}+\mathrm{B}-2\mathrm{C}=3, -2\mathrm{A}+2\mathrm{B}+\mathrm{C}=1 \Leftrightarrow \mathrm{A}=\frac{5}{9}, \mathrm{B}=\frac{4}{3}, \mathrm{C}=-\frac{5}{9}$$

Several theorems are necessary *to validate this technique*, that is, to prove without further checking that $\frac{3x+1}{(x-1)^2(x+2)} = \frac{5}{9(x-1)}+\frac{4}{3(x-1)^2} - \frac{5}{9(x+2)}$. θ_1: two rational functions with the same denominator are equal if and only if their numerators are equal; θ_2: two polynomials are equal if and only if they are of same degree and have the same coefficients; θ_3: theorem about equality of rational numbers; θ_4: theorems about equivalent systems of equation.

[1] Partial fractions: an integrationist perspective. Math 153 Section 55. Vipul Naik. University of Chicago. http://www.indabook.org/d/Integrating-Rational-Functions-by-Partial-Fractions-Using.pdf

Several mathematical theories, "theory" being used in the usual meaning of the word, are necessary to prove these theorems.

Appraisal of the technique: This technique is tedious in some cases without proper software because there are many coefficients to find. In fact, mathematicians are aware of the heaviness of the technique and look for alternatives. For example, another technique consists in plugging in several appropriate values of x depending on the pole order. The technique is based on necessary conditions, so you have to check the equality, unless you have an existence theorem, deriving from a rather extended part of the polynomials arithmetic theory.

Motivation of one step of the technique: Now, as a transition to the corresponding praxeology in automatics, we can ask the following question: why is it important to make the denominator monic? The fact that linear monic polynomials have 1 as a derivative and that the antiderivative of rational functions $1/(x-a)^k$ is easy to calculate is one motive for the restriction to monic factors.

The automatics praxeology for breaking up a rational fraction into partial fractions

The following example is based on a study of how Laplace transform is taught in an on-line course for higher technicians (see Castela & Romo Vázquez 2011 for more details). Hence the mathematics praxeologies have crossed two boundaries: from mathematics to automatic control and then to an automatic course.

Some elements about the automatics' issues are necessary. The problem at stake is automatic regulation of systems: if a quantity is to be kept constant, an electronic gauge measures its value; when variation is recorded, an appropriate regulation process is triggered to go back to the desired value. The less time needed to get the quantity back to this value, the more efficient the control system. The evolutions of the different systems involved are described by differential equations, turned to algebraic ones by the Laplace transform and easily solved, with a rational fraction $F(p)$ as a solution. To inverse the Laplace transform, the online textbook recommends using a table of Laplace transforms. The type of tasks *Breaking up a rational fraction into partial fractions* appears when complicated $F(p)$ are involved. In what follows, I give an idea of the technique and technology proposed by the textbook.

Description of the technique: Assuming that the mathematical techniques are familiar to the students, the author only specifies that $F(p)$ denominator must be written under the following canonical form $k(1+\tau_1 p)\ (1+\tau_2 p)$ … with decreasing values of the τ_i. E.g. $3p+2$ is transformed into $2(1+1.5p)$, not into $3(p+2/3)$. This is a significant change to the mathematical technique.

Motivation (raison d'être) of this special factorisation: If $F(p)=\frac{1}{1+1.5p}$, the corresponding original function is $f(t) = K(1-e^{-t/1.5})$. 1.5 is called the time constant of this function. The system reactivity, and therefore its quality, is directly dependent on the higher value of the time constants. Hence, this value must clearly appear in the calculation. This means that the boundary crossing has changed the type of tasks and thus the technique.

Explanation of the relation between time constant and reactivity: if $f(t)$ represents the controlled quantity and K its desired constant value, it is known that after 7_{τ}, here 7×1.5 seconds, the exponential will be equal to 0, that is, considered as negligible in Automatics. Hence, the transitional regime lasts $7 \times 1{,}5$ seconds.

Validation of this claim: $e^{-t/\tau} < 0.01$, hence $t/\tau > 100$, $t > \tau ln(100) \approx 7\,\tau$.

What needs does the technology of a technique intend to satisfy?

(Castela & Romo Vázquez 2011) analyses the Laplace transform chapter in one mathematics and two automatics textbooks from tertiary vocational courses for engineers and higher technicians. The first one, in a classical mathematics style, is focused on the comprehensive accurate presentation of concepts, theorems and proofs. The Laplace transform technique to solve non-linear differential equations is alluded to, without any examples related to engineering sciences. As shown in the above example, the automatics textbooks are very different. They give a lower priority to mathematical proofs and instead, they develop another kind of knowledge about techniques, strongly correlated with the vocational context. Actually there are many things to know about Laplace transform and the derived techniques, but all these technological elements satisfy diverse needs. Drawing on the aforementioned textbooks, Castela and Romo Vázquez (2011, pp. 88-90) differentiate six of them: *describing* the technique, *validating* it i.e. proving that this technique produces what is expected from it, *explaining* the reasons why this technique is efficient (knowing about causes), *motivating* the different gestures of the technique (knowing about objectives), *making it easier* to use the technique and *appraising* it (with regard to the field of efficiency, to the using comfort, relatively to other available techniques). Such technological elements are present in both previous examples of mathematics and automatics praxeologies. This list should not be taken as exhaustive. For instance, I currently consider one more need: *controlling* the technique implementation by the individuals, that is making sure they have correctly used the technique.

The technological analysis as a relevant tool for transition issues

This approach is an incentive for vocational institutions to analyse the nature and extent of the transpositive effects on mathematics praxeologies within the scientific and professional fields included in their curriculum. As seen in the example above, each component of the praxeology may change or develop for rational reasons that take into account the specific conditions of activities. Educational institutions should consider the motives and legitimacy of these evolutions.

Furthermore, the analysis grid of the technological component is also relevant when the issue of transition to advanced mathematics is addressed. Advancing in mathematics not only consists in learning new theories, it also means facing tasks that get closer to the mathematicians' activities. According to the ATD, any human activity contains elements of genericity. Hence, mathematics researchers, even if they have to be creative, also draw on previously developed praxeologies with a technological component that satisfies practical needs within problem solving and generally derives from experiencing the technique implementation, in other words, not from a mathematical theory. Most of this practical part of mathematics praxeologies is not taught. Yet students need such knowledge, in France at least as of high school (for detailed argumentation, see Castela 2009). So the responsibility for building this practical knowledge lies on the students, and this may be an important cause of failure. Therefore, the challenge for tertiary education would be to organise the students' training to praxeological development from their own mathematical experiences, especially for maths majors.

Modelling the praxeological inter-institutions dynamics

In the foregoing, we have considered one praxeology produced by a research institution in mathematics. The online textbook designed by a college lecturer to teach this praxeology reveals that the technology of the technique contains fragments of knowledge substantiated by mathematical proofs deriving from mathematical theories, as well as other practical elements, empirically developed by mathematicians as they use the considered technique. This part of the technology, being very much linked to the concrete conditions of mathematicians' activities, may appear within a mathematical education setting. However, it will be generally considered of low interest by mathematicians when the dissemination of the mathematical praxeologies to other non educative institutions is at stake. This hypothesis sustains the following modelling of the transpositive effects on a praxeology produced by a research institution and crossing a boundary, that is to say moving from one institution to another in order to be used or taught.

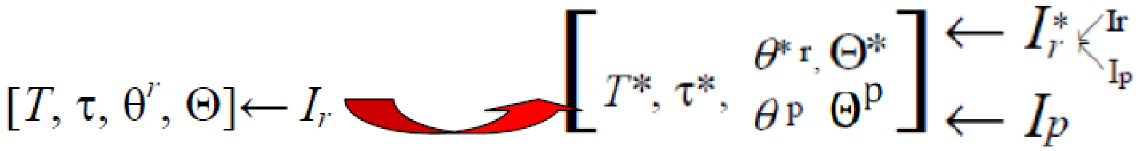

Figure 1. From I_r to I_p, the transposition model

Let us consider the different symbols in this figure which generalises the mathematics case.

The original praxeology

I_r is a research institution, namely an institution socially in charge of producing new praxeologies to address certain types of tasks and organising systematic processes of validation in order to substantiate their legitimacy and institutionalisation. The arrow between the praxeology and the institution represents these institutional processes, which have both an epistemological dimension and a social one.

This research institution may be a scientific one or a technical one. But the category is much extended: the crucial point is that I_r is not directly interested in addressing tasks of the type T. For instance, in the French IREM (*Instituts de Recherche sur l'Enseignement des Mathématiques*), teachers meet to develop collective thinking, design teaching sequences they implement in their classes, assess and if necessary consider afresh. They take some distance from their daily teaching practices and assume the role of researcher.

It should be underlined that the validation processes depend on the specific paradigm of I_r. In the case of mathematics, the technology is proved by demonstrations, sustained by theories that are assessed by the mathematics community. But we know that in physics, the relation between claims and theories is very different, not to mention human sciences and mathematics education.

Finally, let us note that the symbol I_r is a simplification. In fact, a whole hierarchy of embedded institutions are involved in the research activities: laboratories, mathematics journals, congresses, etc.

The transposed praxeology

An institution, some subjects of which have to address tasks of the type T, imports the original praxeology produced and warranted by I_r. In Figure 1, this institution is represented by the symbol I_p underlining the fact that this institution has only pragmatic relations to the praxeology and its development. This I_p (or hierarchy of embedded institutions) could be a research institution of the same domain, using the technique to address other types of tasks (e.g. breaking up a rational fraction to integrate it), or of another domain (e.g. Automatic in the first part of this text). It could also be a professional institution or an educative institution (about didactic transposition, see Chevallard 1985, 1989).

The asterisks express the idea that every component of the original praxeology may evolve. This transformation is an object of institutional transactions completed in a specific institution I^{*}_{r}, created and controlled by I_r and I_p. I^{*}_{r} is more or less vanishing, the transactions are more or less difficult and controversial, depending on several factors: the extent of the transformations (e.g. no transformation if I_p is a mathematics laboratory), the distance between the two institutional epistemologies (e.g., I_r is mathematics and I_p is an experimental science), the importance for I_p that I_r validates the new technique (e.g., if I_p is a profession such as nursing, with high security requirements, it will be of great importance for this professional institution that, the technique remains valid despite the changes), and the importance for I_r that the transposed praxeology not be too far from the original one (in France, it is common for mathematicians to have a critical look on what is taught when I_p is an educational institution).

At last, this diagram says that a practical technology θ_p is developed and acknowledged by I_p on specific empirical bases, possibly sustained by a discourse of second level, which according to the ATD is considered as a theory. Hence the symbol Θ^p represents a true oxymoron, a pragmatic theory. Such object has not yet been thoroughly investigated within ATD framework; Castela and Elguero (2013) suggest that the technologies of the validation praxeologies developed in the institution I_p could contribute to such theory. However, the extent of this theoretical discourse depends very much on the institution's nature and may be quite limited. Let us notice that this model puts forward, not only the institutional knowledge formulated in the technological and theoretical discourse, but also all the social processes of validation and acknowledgment (represented by the arrows) which represent objects of interest for an anthropological epistemology of institutional praxeologies. For instance, in his PhD dissertation, Morel (2013) shows that during the second half of the 18th century, the mining administration in Saxony creates a mining academy in Freiberg to train the mine officials and imposes, from 1797, that a mathematics teacher becomes in charge of a geometrical course for surveyors (*Markscheider*), whereas before, qualified surveyors were responsible of the teaching of the practical praxeologies for mining, within the training program. Morel substantiates that, at that time, the change does not take place smoothly. Such an educative decision from this professional institution is a social acknowledgement of the geometrical praxeologies produced by mathematicians. We may assume that organising the dissemination of a given praxeology relating to a type T for the subjects concerned by T is one of the more primitive and more frequent ways of institutionally acknowledging the legitimacy of this very praxeology.

Other dynamics

So far, we have only considered a specific case, namely a praxeology produced by a research institution, moving to another institution in order to be used or taught. Yet, even with the very broad meaning given to the notion of research institution, we cannot assume that the praxeological production is exclusively operated in such institutions. Occupational and, more broadly, social settings may develop their own original praxeologies, $[T, \tau, \theta^p, \Theta^p]$, within the empirical context of working and of social life. These praxeologies move to other institutions in order to be used with possible transpositive effects. Another possibility is that a research institution, created by the occupational one or not, investigates $[T, \tau, \theta^p, \Theta^p]$ with the objective of improving it and more systematically substantiating its validity. As an example, refer to Castela and Elguero (2013) who examined the case of custom dressmaking in Argentina, with various sized institutions involved. We won't detail this situation any further within the limits of this text.

Conclusion

In this paper, I have addressed the issue of transitions between secondary and tertiary education and between general and professional oriented programs from an epistemological point of view. This proposal is centred on the notion of praxeology, an ATD key concept, with two directions: a grid to investigate the technology of a technique, based on an analysis of the different needs created by the technique used in a given institutional context and a model for the transpositive effects of inter-institutional praxeological dynamics. I think that the first tool is especially interesting for mathematics majors' higher education. In fact, it introduces an organisation of mathematical knowledge that gives as much importance to types of problems and techniques as to concepts and theories. It also acknowledges the necessity of some mathematical practical knowledge as a component of the mathematicians' expertise. Hence, it may support the design of modules aiming at training the students not only to solve problems, but also to draw fragments of this practical knowledge from their experience as well as from the proofs given in lectures.

Furthermore, I believe that the second part of the text is a relevant tool for addressing the issue of choosing the appropriate mathematics for professional oriented higher education. To tackle this problem, mathematicians need to take some distance with their own culture, with their mathematical *alma mater*, as Chevallard is wont to say. They have to reconsider the following questions: which mathematical praxeologies are useful for such engineering or professional domains? What needs would be satisfied? Which discourse makes the mathematical technique intelligible? This is actually an epistemological investigation that we consider as a prerequisite to the design of mathematics syllabi for professional training programs. Mathematics researchers and lecturers are too often not aware of the necessity and of the complexity of such an investigation; they are not necessarily prepared for it by their mathematics education. This should be accomplished collectively with researchers and professionals of the domains using the mathematics at stake in the program. I assume that the text's proposals could first introduce this epistemological problematic and then support the investigations.

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The theory of banquets: epistemology and didactics for the learning and teaching of abstract algebra

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I will sketch in this short communication the main features of an epistemological, didactical and cognitive framework for algebraic structuralism together with an activity, the "theory of banquets" (a "banquet" is an invented structure simpler than group theory, still quite rich semantically), which has been designed using the methodology of didactical engineering. This activity aims at operating the fundamental concrete-abstract and syntax-semantic dialectics and at clarifying the meta-concept of mathematical structure using the meta lever. Empirical results obtained from a classroom realization of the engineering and also laboratory sessions will be discussed. These investigations lead to a better understanding of students' difficulties in abstract algebra which are inherent to structuralist thinking.

Epistemological, didactical and cognitive framework for algebraic structuralism

The structuralist algebra is the result of the abstract mathematics that developed in the first third of the 20th century in the German school of Hilbert and Noether. Classical algebra has been completely rewritten in terms of abstract concepts:

> *This image of the discipline turned the conceptual hierarchy of classical algebra upside-down. Groups, fields, rings and other related concepts, appeared now at the main focus of interest, based on the implicit realization that all these concepts are, in fact, instances of a more general, underlying idea: the idea of an algebraic structure (Corry, 2007).*

In other words, the systematization of the axiomatic method by structuralists led to the **vanishing of concrete mathematical objects in favor of hovering abstract structures**. This induces the following didactical problems: the teaching of abstract algebra tends to present a **semantic deficiency** regarding mathematical structures, which are defined by abstract axiomatic systems and whose syntactic aspects prevail. How does the learner build an "abstract group concept"? Indeed, what kind of representations can he rely on to do so when the purpose is to discard the particular nature of elements, in other words the mathematical context? Moreover, the investigation of the didactic transposition of the notion of structure shows that it is a *meta-concept* that is never mathematically defined in any course or textbook (and cannot be so at this learning stage):

> *As a consequence, students are supposed to learn by themselves and by the examples what is meant by a structure whereas sentences like "a homomorphism is a structure-preserving function" is supposed to help them make sense of a homomorphism (Hausberger 2013).*

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings*. Kassel, Germany: Universitätsbibliothek Kassel.
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French philosophers Lautman and Cavaillès have carefully analyzed the thought processes engaged in structuralist thinking in terms of **fundamental dialectics** (form and content, concrete and abstract) and **two movements**, **idealization** and **thematization**, that operate transversely and lead to **different levels** of objects-structures, as structures may themselves be taken as objects (see Hausberger 2015b for details). Our purpose is to turn these epistemological dialectics into didactical dialectics. Moreover, according to Bourbaki (1950), "each structure carries with it its own language, freighted with special intuitive references derived from the theories from which the axiomatic analysis has derived the structure". This supports Freudenthal's thesis that **structures organize phenomena** and are connected to **mental objects**:

> *Our mathematical concepts, structures, ideas have been invented as tools to organize the phenomena of the physical, social and mental world. Phenomenology of a mathematical concept, structure, or idea means describing it in its relation to the phenomena for which it was created, and to which it has been extended in the learning process of mankind (Freudenthal 1961).*

We shall consider, in abstract algebra, two levels of phenomena/mean of organization:

- the level of **the structure** (group, banquet) which organizes phenomena related to objects of lower level (principle of organization 1, **PO1**)
- the level of **the meta-concept of structure**, which plays an architectural role in the elaboration of mathematical theories using the structuralist methodology (**PO2**)

PO1 is related to the movement of abstraction-idealization and implies a dialectical relation between the structure and the objects that it formalizes, generalizes and uniformizes. On a syntactical point of view, abstraction-idealization amounts to isolating the formal properties of the relations between objects in order to produce the system of axioms that describes the "logic of relations". On a semantic point of view, abstraction-idealization leads to the identification of all the models (in different domains of phenomena) that share the same structure, that is which are isomorphic (principle of abstraction on the basis of a relation of equivalence). Therefore, the **concrete-abstract dialectic** involved in abstraction-idealization is carrying a **syntax-semantic dialectic** and the isomorphism classes appear as intermediary between the semantic domain of concrete objects and the syntactical abstract structure. The price to pay is the transition from objects to classes. We argue that the reification of these classes, that we call structural objects, is important in the conceptualization process of an abstract structure, in a similar manner as standard mathematical objects are conceptualized by the mean of different semiotic representations which refer to the same object since they may be related to one-another by suitable transformations preserving the object. These structural objects should be related to mental objects that may be investigated through the semiotic representations produced by a learner engaged in the task of classification of models.

PO2 is concerned with abstraction-thematization and requires a meta-cognitive point of view on the first movement of idealization. The relational point of view is extended from the elements (a structure encodes relations) to sets of elements through the notion of homomorphism (whose kernel defines distinghished sets) and leads to a combinatorial of structures (sub-structures, quotients, products of structures, canonical decompositions into simple sub-structures, in other terms structural theorems). Understanding the structural dimension of abstract algebra thus requires the consideration of different structures and some form of reflexive thinking.

The theory of banquets

Our approach bears some similarities with the exercises of mathematization elaborated by Steiner at the secondary school level in the context of the New Math reform (Steiner 1968) and his “spiral approach” to algebraic concepts. It is also inspired by Freudenthal's didactical phenomenology of mathematical structures. As a piece of **didactical engineering** (Artigue 2009), it relies on the epistemological analysis of algebraic structuralism sketched above and on Brousseau's theory of didactical situations (Brousseau 1997).

A **banquet** is a set E (the objects) endowed with a binary relation R (encoding the relations between objects) which satisfies the following axioms: $A1$. No object fulfills xRx $A2$. If xRy and xRz then $y = z$ $A3$. If yRx and zRx then $y = z$. $A4$. For all x, there exists at least one y such that xRy. The banquet structure benefits from a very interesting diversity of models that may be constructed in a diversity of mathematical frameworks using different semiotic representations: an empirical setting (“wedding banquets”), models built in set theory or using matrix theory, graph theory, function theory or permutation theory. One framework may be more adequate than the other depending on the task and different treatments may be described to convert from a setting to another (see Hausberger 2015a).

As an activity, the banquet theory is divided into three main sub-activities: the construction and classification of models, an activity of definition by axioms of “tables” (an abstract characterization of the configurations of people sitting around round tables), and an activity of theoretical elaboration (definition of a sub-banquet, an irreducible banquet, the banquet generated by an element, and finally the statement and proof of the structural theorem of banquets of finite cardinal: any finite banquet is the disjoint union of tables). These sub-activities aim respectively at developing among learners the semantic aspects of abstraction-idealization, its syntactical aspects and finally the process of abstraction-thematization. The full banquet worksheet is available in Hausberger (2015b) and the first sub-activity is discussed in Hausberger (2015a).

Meta discourse (for instance the relational point of view provided by abstract systems of axioms) is introduced in the milieu throughout the worksheet. The adaptation of the learner sought for in this game against a milieu dedicated to the learning of structuralist thinking implies **reflective abstraction**: the analogy between banquet theory and group theory shall entitle the learner to thematize such notions as a sub-banquet or an isomorphism of banquets, for instance, and facilitate the task of classification of models. A finite banquet is nothing but a permutation without fixed points and isomorphism classes of banquets correspond to conjugacy classes of permutations, which explains why the theory of banquets is mathematically interesting yet wouldn't be found in any textbook. Nevertheless, this analogy is quite hidden since a binary relation appears quite different from a composition law. Our point is also to allow the development of a mental image of union of circles underneath the banquet structure (which correspond mathematically to the canonical decomposition of a permutation into cycles), in order to implement Freudenthal's idea.

Summary of empirical results

A classroom implementation of the banquet activity has been carried out at Montpellier University with third year students having a background in group theory before teaching ring and field theory. Students' preconceptions regarding the meta-concept of mathematical structure had previously been collected through a questionnaire. Interestingly, our data (Hausberger 2015b) confirm an interrelation between the state of development of the meta-concept of structure and the ability to accomplish the tasks proposed in the banquet activity. For instance, students who do not clearly distinguish the level of objects and the level of structure develop a semi-empirical banquet theory or classify banquets using syntactical methods without articulation of syntax and semantics, whereas the integration of both principles OP1 & OP2 seem to allow students to use the organizing dimension of concepts in the proofs. These findings support the pertinence of our approach as a lever for the learning of abstract algebra.

The classroom sessions with small groups of students and two laboratory sessions with pairs of students both illustrated the important role played by the phenomenological mental image of wedding banquets as an anchor point to syntactical reasoning on the axioms and to the classification task. Different levels of intertwining between this mental image and mathematical symbolism are visible in the students' procedures, in particular the introduction of graph theory for a synthetic representation of relations and the recognition of isomorphism classes through visual patterns.

Laboratory sessions also shed light on a persistent erroneous conception in the abstraction-idealization process induced by the algebraic symbolism: the banquets defined by xRy, yRz, zRx and xRz, zRy, yRx are considered as two different classes within an abstract classification because algebraic symbols may represent any element and the relation is unspecified. Overcoming this obstacle requires the isomorphism concept as a one-one correspondence which coordinates related elements. The mental image of a permutation of people around the table and the extension to a coordination of "chains" of related elements helped students to build a concept of isomorphism on its etymology of structure-preserving transformation, in coherence with visual patterns.

Obstacles related to abstraction-thematization were also identified: for instance, working out the analogy between the cyclic banquet on the set $\boldsymbol{Z}/4\boldsymbol{Z}$ and the group $(\boldsymbol{Z}/4\boldsymbol{Z},+)$ requires to abstract the type of the relation and focus on the process of its iteration, which is conceptually demanding. Group Theory may appear both as an anchor point and as an obstacle.

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Towards the reconstruction of reasoning patterns in the application of mathematics in signal theory

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This contribution discusses a possibility for conceptualizing didactically relevant aspects of advanced mathematical subject matter in such a way that fits with subject scientific categories considering mathematically learning as a societal mediated process according to Holzkamp (1993). Our concern requires analyzing subject matter, teaching and "university" not as conditions that cause reactions but as meanings in the sense of generalized, societal reified action possibilities. In particular we are arguing on a view reconstructing cognitive relevant aspects of recognition rules (in the sense of Bernstein). Such aspects could inform the analysis of specific selection processes between discourse possibilities and the difficulties students may have in recognizing whether a task has to be understood as a mathematical or an electrotechnical one.

Introduction

This paper contributes to an ongoing major research project that describes and analyzes form and content of advanced mathematics and its teaching and learning from a "subject scientific" point of view. This approach grounds in the so called "Critical Psychology", worked out in Holzkamp (1985), see also Tolman (1991). Recently this theory becomes internationally more known within the mathematics education community due to Roth & Radford (2011), who value "German Critical Psychology" as a further development of the culture-historical approaches by Leontjev (1978) and Vygotsky (1978).

The main features of "Critical Psychology" and its subject scientific point of view are well elaborated psychological categories for describing and analysing subject related experiences, in particular thoughts, actions and learning, in such a way that major societal aspects are inherently incorporated. Within this framework there is so far not much (if any) research done that relates to mathematical learning in the context of higher education.

The structure of the paper is organized as follows: At first we sketch concepts from the subject scientific theory that are relevant for an embedding of our observations concerning the use of mathematics in signal theory (Hochmuth & Schreiber, 2015b); in particular we describe shortly the subject scientific concepts of meaning and reasoning patterns. Then we demonstrate in the following, how elements of Bernstein's theory could contribute to working out basic facets of meanings and reasoning patterns. A short outlook concludes the paper.

Subject Scientific Approach ("Critical Psychology")

Critical Psychology claims to present a scientific discussable and criticizable elaboration of basic psychological concepts (categories). The starting point is a historical-empirical investi-

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gation of general historical-specific characteristics of relations between societal and individual reproduction.

Within the context of this paper there are two important points to notice: First, the actual historical-specific form of subjectivity is characterized by the so called “possibility relation” with respect to the societal reality, which gives and includes in particular the basic experience of intentionality and makes consciousness to a prerequisite for the societal reproduction. Second and connected to the first, the specific modality of subjective action experiences comprises a certain discourse form (“I” speak about my “own” actions in terms of subjective reasonable actions and of premises in the light of “my” living interests.) that characterizes to some extent the specific subject scientific standpoint.

According to this modus, world conditions are given in terms of meanings in the sense of generalized societal action possibilities. Meanings of reality aspects, which are relevant for “me”, become premises. Consequently, subject scientific considerations are essentially given by premises-reasons-relations.

In “Critical Psychology” meanings and their mediation role do not only represent social-interactive but also societal aspects grounded in relations between production and reproduction. Via meanings, human activities, like teaching and learning, can be thought as societal mediated. An analysis of subject activities regarding its societal mediation requires an adequate conceptualization of the objective situation of the subject. As representations of subjectively relevant objective conditions they have to be describable and analyzable as generalized action possibilities, hence meanings.

Since meanings appear (via objective-subjective premises) to some extent as the medium within which subjective action reasoning is grounded, their study is a prerequisite for describing and analysing related cognitive, motivational and emotional processes as aspects of subjective activities like learning under concrete societal conditions.

Meanings in the indicated sense are relevant for acting and thinking, but do not determine them. Furthermore, they are not only of linguistic-symbolic nature, but objective-societal objects, to which symbols relate. Of course, in particular in mathematics and science, symbols are objects by their own and acting with them underlies rules that are epistemologically and institutionally determined and are also determined as elements of a scientific or pedagogical discourse.

The Issue “Recognition Rules”

Embedded in the presented subject-scientific approach we applied (amongst others) some basic ideas from Bernstein’s (1996) theory (in particular the concepts classification, recognition- and realization-rules) for the reconstruction of how students handle the different mathematical praxeologies in solving typical signal theory tasks. Within our framework the goal is to figure out so called reasoning patterns and in particular the conscious-unconscious setting of premises. An important dimension in those analyses is the differentiation between ostensive and conceptual thinking and its relation to recognition-rules. Following Holzkamp (1985) the opposition between ostensive und conceptual thinking represents (to some extent) the historic-specific societal character of the cognitive aspect of human actions, that is the concretization of the general historic-specific opposition between restricted and generalized action potence with respect to “thinking”.

According to our observations (Hochmuth & Schreiber, 2015a) about the epistemological-philosophical background of the use of mathematics in signal theory and corresponding analyses within the framework of the Anthropological Theory of Didactics, the students have somehow to accept a certain type of inconsistencies and to “learn” that they should neglect specific aspects from discourses, for example they have to ignore concept definitions and at the same time, they have to realize aspects from them, for example specific parts of “concept images” (Hochmuth & Schreiber, 2015a, b). In other, Bernstein’s (1996) words: The students have to understand the context specific principles of knowledge classification (recognition rules) and to apply correctly related “realization rules” in view of solving tasks.

Our further empirical investigations show that sometimes tasks send “wrong” signals, that is, they look like mathematical tasks from higher mathematical courses for engineers. Novices in the field then try to apply arguments and techniques from the mathematical discourse but often fail in solving the tasks because of the arising complexity. It requires time and experiences until the students recognize that the electrotechnical discourse establishes some different but subject dependent more efficient techniques (realization rules) that leads to more satisfactorily results. Recognition and realization rules are obviously in strong relation to the selection of premises as well as to contents and structure of reasoning processes.

An Example from Signal Theory

A typical signal theory task looks at follows[1]: *Let be given a low-pass-bounded signal $s(t)$ with Fourier transform $F\{s(t)\} = \begin{cases} 1, & \text{for } |f| \leq W \\ 0, & \text{for } |f| > W \end{cases}$. Considering $s(t)$ as input signal classify the following assertions as true or false and justify your answer: i) The band-with of the output signal generated by a linear system is always less or equal to W. ii) The band-with of the output signal generated by a linear time-invariant system is always less or equal to W.*

Ad i) A solution based on the premise that the task is to understood as mathematical looks as follows:

The system function $h(t) = \cos 2\pi f_0 t = \dfrac{e^{i2\pi f_0 t} + e^{-i2\pi f_0 t}}{2}$ gives for $s(t)$ by

$F\left\{e^{i2\pi f_0 t}\right\}(f) = \delta(f - f_0)$ and

$$F\left\{e^{i2\pi f_0 t} \cdot s(t)\right\}(f) = F\left\{e^{i2\pi f_0 t}\right\} * F\left\{s(t)\right\}(f) = \int_{-\infty}^{\infty} \delta(\tau - f_0)\chi_{[-W,W]}(f - \tau)d\tau = \chi_{[-W,W]}(f - f_0),$$

the output signal with spectrum $\frac{1}{2}\left\{\chi_{[-W,W]}(f - f_0) + \chi_{[-W,W]}(f + f_0)\right\}$, which represents a signal with a shifted band-with. Hence the assertion i) is false.

[1] The authors are grateful to Prof. Dahlhaus (University of Kassel) for placing the task and student solutions at our disposal.

An answer that interprets the task as signal theoretical would be: In a general linear system the transfer function changes in time, which could induce new frequencies and change the band-with.

Ad ii) "Mathematical solution": For a linear and time-invariant system the output signal of the input signal $g(t)$ is given by

$$F\{L(g)(t)\}(f) = H(f)F(g(t)\}(f).$$

This relation implies that the assertion is true, since supp

$$F\{L(g)(t)\} \subseteq \text{supp } F\{g(t)\}.$$

"Signal theory solution": Since the transfer function does not change in time, no new frequencies arise.

Checking the approaches presented in the course lectures and the course material we expected in part ii) that both types of solutions arise in the students' homework. In fact mainly solutions that fit into the mathematical-type were given. In the subsequent observational study also signal-theory-type solutions were presented.

All reasoning trials to part i) are of mathematical type. While the students' homework solutions are far from being correct and complete, the answers in the observational study (shortly after the exam) were mainly correct. Despite the complexity of the mathematical-type solution no student tries a signal-theory-type solution.

Outlook

In a next step one has to reconstruct typical premises-reasons-connections as part of the general causes-reasons-connections including the differentiation between ostensive and conceptual thinking, which will enclose further empirical research questions regarding tasks and solution processes and where among others video and interview data will be used.

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In-depth interviews as a tool in didactics of mathematics

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We aim at characterizing what mathematical experts and novices think about scientific learning and research: about the epistemology of their subject, about relevant skills and about beneficial attitudes, as well as biographical features (dis)advantageous for prolific scientific work. To gain significant qualitative data we adopt techniques for in-depth interviews that include associative and projective methods stemming from systemic and person-centred counselling. One long-term goal is to obtain evidence-based recommendations on how to establish adequate propaedeutics in secondary and tertiary science education in the STEM fields.

Person-centered methods in didactics

Why is it that mathematicians do mathematics? And how do they conceptualise what they are doing? To an outside observer, the behaviour of researchers in, say, pure mathematics might be somewhat puzzling: they struggle with mathematical problems, sometimes for weeks or months, sometimes all by themselves, and mostly with intangible outcomes at best.

A detailed and evidence-based answer to such questions might have consequences for how we teach mathematics, especially on university level, and how we counsel our students with respect to their learning on a cognitive, meta-cognitive as well as emotional level. But there appears to be very little known about these questions in the bold generality in which we have formulated them, apart from occasional anecdotes there seems to be mainly the study of Burton (2004).

There are several equally legitimate points of view on this type of questions, most notably the psychological, concentrating on the individual and what she experiences, and the sociological or ethnographic, focussing on mathematicians as a group with shared narratives and a particular "culture". From both points of view, a quantitative large-scale study[1] should be preceded by a phase of thorough exploration to provide the necessary categories and the necessary theoretical background, and we suggest that person-centered methods are particularly well-suited for this task.

Person-centered methods have a long history in both fields, psychology and ethnography: Person-centered interviews provide a way to treat the interviewee not only as an "informant" who might be asked why he thinks mathematicians do mathematics, but also as a "respondent" who is him- or herself an object of study – observed as she speaks as freely as possible about her own relevant experiences, beliefs and attitudes (Levy and Hollan 1998, Rogers 1957, Langer 2000).

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What makes person-centered methods interesting in a didactical setting is the fact that they lend themselves quite easily to being applied in counselling and learning environments (Cornelius-White & Harbaugh 2007). In this presentation, we will highlight person-centered methods as both, empirical and didactical tools.

The importance of narratives of scientific learning and research

A central part of person-centered interviews is the choice of questions and stimuli designed to help the interviewee to make contact with her inner experience and to freely talk about relevant events, attitudes, beliefs, feelings etc. For us, the most relevant aspects are the internal self-concept of the interviewee as a scientist and how he or she experiences the process of doing mathematics.

In the case of scientists and their individual motivation, there is some evidence that the self-concept is of paramount importance compared to other motivational factors. Indeed, a study of James C. Ryan (2014) on the work motivation of UK-based research scientists (N = 405) working in the chemical, biological and biomedical research domains suggests that *internal self-concept motivation* is a key factor for the work motivation of scientists; this source of motivation is one of five compared in the study and "represents an individual's motivation to adhere to their [sic!] internal standards of traits, competencies and values". The other four are (in descending order of their measure of importance) *goal internalisation motivation*, *intrinsic process motivation*, *external self-concept motivation*, and *instrumental motivation*.

These rather general findings do not clarify what the self-concept of a mathematician might actually look like (and it is of course likely that it is not stable over time). The self-concept might be approached by analysing the narratives that scientists tell about their learning and their research. Of course, one should not necessarily take these narratives at face value, but they are of ethnographical interest in their own right and might serve as a starting point for an interview that opens up a space for the interviewee to talk about his or her inner experiences more deeply.

Narratives of scientific learning and research

One type of narrative that can be found in preambles and other parts of school and university curricula is the narrative of mathematics as an instrument to achieve goals that are outside of mathematics, e.g. mathematics as a tool to be used in other sciences or as a means of enhancing argumentative competencies to be employed in other contexts.

We can supplement these narratives with dissenting stories that seem to be highly relevant for the self-concept of prolific scientists.

An archetypical example

Richard Feynman, the world renowned theoretical physicist, gave the following account of his attitude towards his research at Cornell University; there, he had taken up a job in the late 1940ies after having spent several years at Los Alamos, working diligently on the construction of the atomic bomb (Feynman 1985, emph. In original):

> *Then I had another thought: Physics disgusts me a little bit now, but I used to enjoy doing physics. Why did I enjoy it? I used to play with it. I used to do whatever I felt like doing – it didn't have to do with whether it was important for the development of nuclear physics, but whether it was interesting and amusing for me to play with. [...]*
>
> *So I got this new attitude. Now that I am burned out and I'll never accomplish anything, I've got this nice position at the university teaching classes which I rather enjoy, and just like I read the Arabian Nights for pleasure, I'm going to play with physics, whenever I want to, without worrying about any importance whatsoever.*

Feynman went on to not only ponder upon the physics of how flying dinner plates wobble when rotating, but he also did the work on quantum electrodynamics that finally won him the Nobel Prize.

Feynman was, in many ways, not the typical scientist. But this story presents us with an archetype of the kind of rhetoric many scientists might use to describe their perspective on research: The allusion to the polarity of work and play, combined with the expressed belief that this polarity is not a complete description of what is going on in research as play is supposed to be an integral part of how a (pure) scientist might actually work.

Of course, also Feynman would surely admit that play is not the only way to do research – after all, he has been part of the Manhattan project, an endeavour where all means were directed to a single goal, the atomic bomb, and no bit of research was supposed to be an end to itself. But to us, it is an interesting question where eminent scientists position themselves with respect to the relation of work and play in research and how their attitude towards play is related to their very eminence.

Rhetorics of play

The narrative of play that we have just met is surely worth to be analysed a little further. In his seminal book "*Homo Ludens*" ("Man the Player"), Johan Huizinga paints the image of play as a driving force behind most if not all human culture (Huizinga 1955). Building on his work, Roger Caillois provides the following, now classic characterisation of play (Caillois 2001):

Play is free, not obligatory; it is separate – circumscribed within limits defined and fixed in advance; it is uncertain, and some latitude for innovation is left to the player's initiative; it is governed by rules; it is accompanied by a special awareness of a second reality; it is unproductive in that it creates no wealth and ends as it begins.

If you replace "play" with "pure scientific research" in this definition then one could argue that you get a reasonably good description of the kind of research Feynman seems to have had in mind (at least if you interpret "unproductive" as "not being intended to be productive").

A more subtle and more diverse image of play can be obtained by considering various *rhetorics of play* as was done by Brian Sutton-Smith (1997) – he lists seven of them. Feynman's narrative evokes what Sutton-Smith calls the *Rhetoric of Self*: it interprets "play in terms of subjective experiences of the player [...]; it is an optimal experience, an escape, a release; it is intrinsically motivated [...]." This rhetoric has its natural counterpart in a *Rhetoric of Frivolity* that denigrates play as a "waste of time, as idleness, as triviality, and as frivolity", rooted in what might be called a puritan work ethic. The opposition of work and play can itself be analysed as a rhetorical figure and appears to be closely connected to Western culture. So putting *Homo Ludens*, in opposition to *Homo Faber* ("Man the Worker") is itself part of a narrative.

Methods to gather narratives and to facilitate self-clarification

Person-centered interview methods

"Person-centered interviews are a mixture of informant and respondent questions and probes. A probe is an intervention to elicit more information, not necessarily in the form of a question." (Levy and Hollan 1998). The corner stones of person-centered methods are the accepting attitude of the interviewer towards the interviewee and their relationship (Rogers 1957, Langer 2000). "Probes" that might be used in a person-centered interview on the self-concept of mathematicians could comprise:

- A "mathematical fever chart": The interviewee is asked to map his "mathematical biography" on a sheet of paper, as a graph that resembles a fever chart; the time frame could include the time spent at school and university. The particular meaning of "high" and "low" can remain somewhat ambiguous when the task is assigned to the interviewee, the idea being to leave as much space as possible to the interviewee. The interviewee is then asked to name and describe critical points of the graph, giving the interviewer some insight in some emotions and narratives connected to relevant events in the interviewee's biography.
- A collection of narratives is offered to the interviewee. This could take the form of short texts, each on a single card, or even of drawings or other images. The narratives could be taken from Sutton-Smith's list of narratives of play, a list that could actually be read as a list of ways to (consciously or unconsciously) justify acts in general, together with ways to denounce certain acts as immoral or frivolous. The interviewee is now asked to choose cards that he or she considers relevant (for example in light of the biographical information provided in the chart described above). She can now elaborate on how she relates to the narratives on the cards and how her own narratives differ. It is important to include some blank cards to allow the interviewee to substantiate further narratives; more confident interviewees can be asked to draw their own drawings on a card, turning this method into a projective method.
- To increase the authenticity of the situation, the interviewee might first be asked to attack a brain teaser (e.g. a mathematical problem) and to comment on it. It has become apparent that mathematicians quite often react quite strongly and emotionally on mathematical problems that they consider interesting (for further information and a more thorough didactical analysis see (Friedewold & Nicolaisen & Schnieder 2015)). This method could replace the "mathematical fewer chart" in that it provides occasions to talk about narratives of doing mathematics.

These and further person-centered methods can be used in an interview setting – ideally, not only is the audio of the interview transcribed afterwards, but also the relevant paraverbal and nonverbal communication is monitored.

Quite obviously, such an interview can be transformed into a learning environment to enhance the self-awareness of the client by simply replacing the roles of interviewer and interviewee with the roles of teacher/counsellor and student/client.

A pedagogical workshop

The workshop we have planned together with Frauke Link, HTWG Konstanz, is aimed at mathematics lecturers who are ready to examine their mathematical biography. The workshop proceeds on two different levels: First, we try to offer the participants a framework for biographical self-clarification and to ponder on the question: "Why do I do mathematics?" Second, we want to investigate whether the participants' answers contain objectifiable narratives.

The twofold objective of the workshop raises a dilemma: the dilemma between free exploration and structured self-examination. The workshop wants to offer help in a very personal area of self-clarification, in which every participant can explore him- or herself freely and to the depth of his or her choice; the categories to describe this exploration should be found, formulated and explored autonomously.

The subjective diversity that is associated with free self-exploration could of course be avoided by a structured self-examination, in which the categories of self-clarification and self-description are specified in advance and thinking and perception are thus objectively channelled, for instance in the course of a very explicitly structured interview. There does not seem to exist any guideline, any universally approved principles, factors or terminology on which autobiographically oriented attempts to examine oneself could be based and that could be used to separate significant, i.e., transforming experiences in the transition to being a mathematician from "Erfahrungskitsch" ("experience kitsch", Mollenhauer 2008).

The didactical point of our workshop – our attempt to bypass this dilemma – consists in essentially reducing the question "Why do I do mathematics?" to the examination of the following five groups of questions:

- Where did you first encounter mathematics unbraked/unretarded? Which values and which social role was *presented* to you in the process?
- To you, how was mathematics *represented* in school/university? How was mathematics communicated and taught to you there?
- How self-determined were you in developing and contributing your mathematical interests in school/university? When did you become the autonomous subject of your learning of mathematics?
- How much was expected of you as a student? Could it have been more? When was the first time you solved mathematical problems autonomously?
- Do these experiences still have an impact on your mathematical identity today? To what extend?

The point of these five groups of questions is their orientation towards general pedagogical theories (Mollenhauer 2008, Benner 2012). According to these approaches, upbringing and education ("Bildung") are constitutive for how human beings become humans, i.e., for what then realizes itself as (mathematical) personality and identity. Hence, personality as a result of educational processes can be reconstructed as the integration of experiences of an individual with presentation, with representation, as an autonomous subject of its own learning ("Bildsamkeit"), with self-regulated learning ("Selbsttätigkeit") and with identity.

In this respect, they span the elementary topics of the process of biographic self-assurance. If, therefore, Kant (1900) states in the introduction of his lecture on education: "Man can only become man by education. He is merely what education makes of him. It is noticeable that man is only educated by man—that is, by men who have themselves been educated.", then we conclude from this: human identity forms itself in relation with and delimitation of educational processes faced by the individual. And these processes have an objectifiable basic structure that we try to capture by the five groups of questions that we have given above.

Given the generality but limited number of these questions, we remain this side of a semi-structured interview; we thus avoid the perspective on the individual plurality of biographical formations to be prematurely constricted by predetermined categories. Then again, we should not present the abovementioned questions without comment: we make it clear that the coarsely prestructured questions are meant to partially release the participants from preliminary conceptualizing and analyzing.

First categories found in an explorative study

At a conference on tertiary mathematics education, Frauke Link conducted a workshop, along the above-mentioned lines, with mathematicians (N = 15) from several universities in Germany who had varying degrees of experience as lecturers or teachers. The qualitative (written) data that we gathered at the workshop was complemented by three in-depth interviews. From this data, we have extracted a first list of categories that we wish to refine by further qualitative research:

1) *Applications to the real world:* This narrative is characterised by the allusion to real world or scientific applications of mathematics that lie outside mathematics itself. The mathematical language or mathematical results are highly relevant for other fields such as physics, engineering etc. and the relevance of mathematics lies in both, how it increases our capability to understand the world and how it enables us to design products such as computers, cars, etc. Typical examples of statements include "mathematics is the language of reality".

2) *Reliability of logic:* The characterizing feature of this narrative is that doing mathematics is perceived as something positive because it allows the mathematician to take part in a pure and reliable world. This world is experienced as being supportive, clear, reliable and therefore enjoyable. One is lead to contrast it to the real world which is perhaps perceived as unreliable, unpredictable or obscure. An archetypical example is reproduced below ("I sense happiness when I reconstruct/comprehend proofs and I perceive mathematical structures as capable of bearing.").

Ich empfinde im Nachvollziehen von Beweisen Glück.
und math. Strukturen als tragfähig.

3) *Epistemic curiosity:* This narrative highlights the curiosity of the researcher and her wish to better understand mathematical structures. It embeds into the ancient and venerable narrative of the epistemic curiosity of scientists ("Men pursue science in order to know, and not for any utilitarian end." – Aristotle, cited after Posnock (1991)), but the objects of curiosity have a very particular, abstract form. "Even today, to recognize and to understand structures is my main motivation when dealing with mathematics."

4) *Meanings and bonds:* One interviewee summarized this narrative as follows: He reported that he was fascinated by mathematical problems and by giving meaning to the mathematical contents he encountered to add: "And that was, many times, a love-hate relationship that [...] only develops if you are very closely connected to something. This is, I think, similar as it is in interpersonal relationships."

5) *(Frivolous) play:* This is the narrative mentioned above and exemplified by a quote by Richard Feynman. It was much less prominent than expected when we collected our data in the rather non-directive way described above, but our interviewees could sympathize with it when it was offered to them directly after the actual interview. This may hint to the fact that narratives that may seem socially undesirable are underrepresented in such an open format of inquiry. In a quantitative follow-up to this study, this issue will have to be addressed.

6) *Talent for mathematics:* A narrative that is not typical for mathematics as a science or, in fact, scientific pursuit in general, but can potentially be found in most professions is the narrative of talent. It can take a rather self-determined form ("Because I can do it better than anything else."), but there are variants that allude to a certain lack of autonomy ("It was like sliding into [mathematics] via physics.").

7) *Immediate gratification / flow:* The narrative of mathematical problem solving providing flow experiences (with the solution being immediately gratifying) might seem to be very unspecific at first glance, but there are some special features to it: In school, students might encounter many mathematical tasks that provide clear and immediate feedback (in contrast to tasks in other subjects such as art or history), which could make mathematics gratifying for high-performing students. And you can suspect mathematical problems to allow for flow experiences in a particularly wide range of circumstances (while riding your bicycle, late at night in bed, while painting your walls, etc.).

The audience of the talk kindly suggested two additional narratives that did not show up in our data set so far: *Mathematics as a (competitive) sport* and the *wish to teach mathematics*. The first of the two might be underrepresented as it might have some socially undesirable aspects (at least in a German context). We are curious whether the second of the two will eventually show up as a primary narrative with a sizable number of interviewees, or whether it might have a mostly secondary nature, relying on some other narrative ("... and I would like to pass this on to other people.").

Outlook

Several questions arise naturally from this first part of our study:

- How are the narratives related to the actual experiences of students in school and at university? In what sense might they be "true" and how relevant is this "truth" to the self-image of a person doing mathematics?
- How can we turn this analysis into a valid quantitative tool that can tell us something about the distribution of motivating factors among a population of students?
- If we want our schools and universities to "produce" capable graduates in the STEM-subjects, (how early) should we take the motivational structure of scientists into account?
- Is our list "complete" in the sense that it captures all the relevant narratives?
- If we reframe our research tools as tools of self-clarification, are person-centered methods effective and how much can already be accomplished in a workshop setting?

Next, we are going to focus on the final two questions: A "complete" and accurately defined list of narratives seems to be of value in its own right, and we regard self-clarification as so fundamental for learning a subject that providing effective tools to facilitate it is certainly worthwhile.

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From high school to university mathematics: The change of norms

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There are important differences between high school and university mathematics e. g. with respect to language, modes of argumentation, or ways of presentation. These differences can be regarded as representing different norms in the high school and in the university classroom. The contribution will provide a theoretical embedding of these ideas. Moreover, we will present data, which support the hypothesis that this change of norms is an important challenge for university students.

Transition from school to university mathematics

The transition from high school to university mathematics is challenging for many students (e. g. Hoyles, Newman, & Noss, 2001). Large dropout rates in mathematics, science, and engineering programs demonstrate that many students fail in overcoming the gap between high school and university (Heublein, Richter, Schmelzer, & Sommer, 2014) particularly with respect to mathematics. A major problem might be the change of norms between high school and university mathematics. Norms valid at school are possibly not valid at the university level (Beitlich, Moll, Nagel, & Reiss, 2015), and new norms may emerge in the mathematics class yet unknown from high school mathematics. In particular, norms for using language to express mathematical statements or explanations, norms for arguing mathematically, the norms for presenting results change between high school and university classes.

Changing norms during school and university mathematics

The phenomenon – namely students' problems with mathematics particularly in their first year of university studies – is in principle, well known. We will argue in the following that it can be described based on different theoretical lines. On the one hand, we will refer to Bruner (1966) and his notion of different modes of representation of knowledge and his proposition of a spiral curriculum as a consequence. On the other hand, we will refer to Oser and colleagues and their theory of negative knowledge (Oser & Spychiger, 2005).

With respect to the theories of Bruner (1966), children e. g. at the primary school level will not be able to solve problems in the same way and at the same level as students at high school, but they will probably be able to perform in the same context and get an adequate solution but at a different level. His theory is based on the idea that children are able to learn everything at a level appropriate for their stage of development. This can be illustrated with the example of reflection across a line in the plane. At primary school, the appropriate level might be folding paper, drawing a geometrical figure, or using a geoboard in order to discover the symmetry of a figure. At the lower secondary level, the concept of the reflection is introduced in a more abstract way. Accordingly, students should be able to identify

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the reflection of an arbitrary geometrical object through construction with circle and ruler. At the university level, this concept becomes even more abstract: now a reflection is represented as matrix, and students are supposed to express a reflection as function, to design the matrix, and to calculate a specific image. The methods used at the different stages of development represent valid aspects of knowledge with respect to the underlying concept and the scope of the concept.

Though the aspects may be regarded as representing valid mathematics knowledge, at the university level e. g. paper folding is generally not accepted as solution of a problem. Accordingly, students will not only learn which strategies will solve a problem but whether they are assessed adequate. Understanding the lack of adequacy may be seen as negative knowledge, a concept introduced by Fritz Oser (e. g. Oser & Spychiger, 2005). Negative knowledge is usually described as knowledge of errors, however Gartmeier, Bauer, Gruber and Heid (2008; p. 89) extend it to “nonviable knowledge that is heuristically valid”.

Taking into account these theoretical perspectives, norms for doing mathematics may be regarded as changing in particular between high school and university level. High school mathematics will usually include less formalism. Moreover, it is less important to use the exact terminology. Mostly, the results can be described in everyday language, especially when they are discussed in the classroom. At the university level, the correct use of mathematical language plays an important role, especially in reasoning problems where an exact argumentation is necessary. Also the modes of argumentation are different at the university level. At the high school level, statements are often explained more empirically and narrative and/or with the help of examples. In contrast, at the university level the typical argumentation is driven by an axiomatic-deductive approach and the presentation of arguments is formal. As a consequence, students are not acquainted to high formalism and might struggle with it at the beginning of their studies.

Research Focus

Norms in the mathematics class should particularly address use of language, modes of argumentation, and ways of presentation. It is assumed that there are important differences between high school and university level. Accordingly, answers of first-year students will be analyzed with respect to their (1) correct use of mathematical formalism and terminology, (2) modes of argumentation and, (3) ways of presenting the answers.

Method

To analyze these three aspects, students’ solutions of mathematical reasoning problems were examined. The items were related to the high-school curriculum so that students had the prerequisites for solving them correctly.

Participants

In this study, $N = 439$ ($N = 353$ male) first-year students of engineering (e. g. mechanical engineering, chemistry engineering) took part. The study was part of a transition course in mathematics before the winter term 2014/15 at a university in Germany. Participation in the transition course was voluntary but recommended by the university. The participants’ mean high school grade was 1.7 (*SD* = 0.58), which is above the national average. In Germany, grades vary between 1 and 6 with 1 being the best grade. Students' mean age was 19 years (*SD* = 1.6).

Instrument and Coding

The paper-pencil test consisted of five tasks with four subtasks each resulting in a total of 20 items. Each task encompassed four items related to four school-related geometrical concepts, two of which are introduced at the lower secondary level and two of which are introduced at the upper secondary level: perpendicular bisector of a triangle's side, isosceles triangle, vector, and linear dependence. The items had an open-ended response format and thus the students' answers could be analyzed in depth. The students had 30 minutes for completing the test. The item selection followed a rotation design: every student solved three out of four items of each task. Moreover, mathematical argumentation was tested with a specific task. The results of the analysis of this task are presented in this article. In task 5a *N* = 243 students were asked to prove that the perpendicular bisectors of the sides of a triangle intersect at one point. In task 5b *N* = 287 students were asked to prove the Thales' theorem. In task 5c *N* = 284 students were asked to prove that the addition of two vectors in IR^2 is commutative.

For these three items we analyzed the language, especially when formalism was used correctly (no formalism/ wrong use/ correct use) and when terminology of the concepts was used correctly (no notions/ wrong use/ correct use). The coding "no notions" was given, for example, if students only worked with drawings. Furthermore, the modes of argumentation were analyzed by using a simplified model of the coding scheme of Harel and Sowder (1998): no argumentation/ external/ empirical/ analytical. External argumentation meant that a student was convinced of the truth of the theorem but could not explain why. Here also answers referring to external authorities were included, for example "Because Thales said that". The analytical argumentation contained axiomatic reasoning typical for mathematics. In addition, the way of presentation of results was analyzed (no arguments/ narrative/ narrative and formal/ formal).

Results

The results – according to the three research questions – were highly dependent of the content of the tasks.

Language

In task 5a (intersection of perpendicular bisectors) most of the students did not use formalisms at all (90%). Furthermore, 37% did not use any terminology related to the content, 28% used wrong terminology: for example they used expressions like "hypotenuse" in a triangle, which was not right-angled. In contrast, a third of the students (35%) used correct terminology. For proving the theorem in task 5b (Thales' theorem), 49% used formalisms whereas 10% of them used them incorrectly and 39% correctly. In this task 76% of the students used correct terminology and only 15% wrong. The results of task 5c (vector addition) were similar to them of task 5b: 35% did not use any formalism and 59% used formalism correctly. Only 12% used wrong terminology, 52% correct.

Modes of argumentation

The modes of argumentation were similar in tasks 5a and 5b: About a third of the students (5a: 37%, 5b: 29%) did not argue at all, another third (5a: 28%, 5b: 30%) used external arguments, and another third (5a: 35%, 5b: 39%) argued analytically. Surprisingly, in task 5c even 71% reasoned analytically. 30% argued externally and 29% did not argue at all.

Way of presentation

In task 5a even 57% expressed their answers only with words (narrative). Only 5% used words and some formal expressions like variables or formulas, and also 5% answered only formally. In task 5b 42% of the students argued narratively, 29% narratively and formally, and 20% only formally. In task 5c the results were similar: 33% gave a narrative answer, 29% a narrative and formal one, and 36% a formal one.

Discussion

The fact, that the results differ in the three items, shows that the content of the item was highly relevant. It was obviously important at which school level the content was taught. If content was part of the curriculum only in lower secondary classes, students had more problems in presenting their answers in a formal way. The results of task 5a demonstrated that: Most of the students did not use any formalism (90%) and they also gave mainly narrative answers without any formulas (57%). Content, which had been introduced in the upper secondary classroom (vectors) showed other results: more students used formalism (65%) and gave answers that include formulas or variables (36%). These effects might be explained using the theory of the spiral curriculum: The higher the level on which content was taught, the more familiar students were with formalism and terminology. Accordingly, also students' methodological knowledge develops with concrete examples and cannot easily be transferred to other content. Moreover, the large number of missing arguments (5a: 37%, 5b: 29%) may indicate substantial negative knowledge. Students have a rough understanding what may not be used in the mathematics classroom but are not able to use this knowledge for a (positive) solution of problems. This finding is in line with the one stated above thus confirming this statement in the light of different theoretical considerations.

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Interpretations of equations and solutions in an introductory linear algebra course

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Over the past ten years the IOLA project team has been involved in a series of studies and an ongoing curriculum development project regarding the teaching and learning of linear algebra at the tertiary level in the United States. The study presented here highlights student difficulties in interpreting solutions to systems of equations with emphasis on frameworks to clarify the difficulties and point toward curriculum development aimed at addressing these difficulties.

Literature and Theoretical Framework

The origins of linear algebra lie in efforts to solve systems of linear equations and understand the nature of their solution sets. In our experience, instructors of linear algebra tend to see the work of teaching students to solve linear systems as the more straightforward and procedural portion of the course. We speculate that solving linear systems and interpreting their solution sets in fact entails hidden and significant challenges for students that are important for their later success in linear algebra, as well as their work in related STEM courses. In particular, students will encounter and need to make sense of systems of linear equations that have infinitely many solutions throughout an introductory linear algebra course: for instance when making sense of linearly dependent sets of vectors, when dealing with linear transformations whose null spaces are non-trivial, and when making sense of eigenvectors.

The existence of student struggles in linear algebra is well-documented (e.g., Dreyfus, Hillel, & Sierpinska, 1999; Harel, 2000; Stewart & Thomas, 2009; Larson & Zandieh, 2013). Researchers have speculated that the formalization of ideas such as span, linear independence, null spaces, basis, and eigenvectors is problematic for students for a variety of reasons including their preference for practical rather than theoretical thinking (Dorier & Sierpinska, 2001) and struggles shifting among modes of representation (e.g., Hillel, 2000; Dorier & Sierpinska, 2001). Our own research has focused on the importance of symbolizing (Zandieh, Wawro, Rasmussen, 2015) and the power of making connections across different contexts, metaphors or interpretations for a particular concept (Zandieh & Knapp, 2000; Selinski et al., 2014). In addition our work has pointed to metonymy as a way to describe how we shorten, or condense information in ways that provide efficiency but can also cause confusion (Zandieh & Knapp, 2006).

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In Larson and Zandieh (2013) we developed a framework specific to making sense of student connections across multiple interpretations by identifying three important interpretations of the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ where A is an nxm matrix, $\boldsymbol{x}$ is in $\boldsymbol{R}^m$ and $\boldsymbol{b}$ is in $\boldsymbol{R}^n$. We especially note how the role of the vector $\boldsymbol{x}$ shifts across those interpretations. Namely, $A\boldsymbol{x} = \boldsymbol{b}$ can be interpreted as a system of equations (where $\boldsymbol{x}$ is a point of intersection), a linear combination of column vectors (where $\boldsymbol{x}$ is a set of weights on the column vectors of A), or as a transformation from $\boldsymbol{R}^m$ to $\boldsymbol{R}^n$ (where $\boldsymbol{x}$ is an input vector corresponding to the output vector $\boldsymbol{b}$). In this paper, we expand this framework to the context of augmented matrices – where the literal symbol $\boldsymbol{x}$ disappears completely from the algebraic representation $[A|\boldsymbol{b}]$.

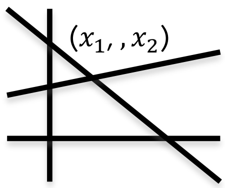
Interpretation of $A\boldsymbol{x} = \boldsymbol{b}$	Symbolic Interpretation	Geometric Interpretation
System of equations interpretation $a_{11}x_1 + a_{12}x_2 = b_1$ $a_{21}x_1 + a_{22}x_2 = b_2$	A: rows viewed as coefficients $(a_{11}, a_{12}, a_{21}, a_{22})$ $\boldsymbol{x}$: solution (x_1, x_2) $\boldsymbol{b}$: two real numbers (b_1, b_2)	
Augmented Matrix interpretation $[A|b]$	A: coefficient matrix entries $\boldsymbol{x}$: Does not appear! $\boldsymbol{b}$: entries of far right (augmented) column	No unique geometric representation

Figure 1. Part of Larson & Zandieh's (2013) framework for views of $A\boldsymbol{x} = \boldsymbol{b}$

Figure 1 illustrates the systems of equations and augmented matrix interpretations in the case where $\boldsymbol{x}$ is a vector in $\boldsymbol{R}^2$ and there is one unique solution. The augmented matrix can be seen as a metonymy for the system of equations. It reduces the information stated for efficiency, but as we will see in the Findings section, this reduction in information can be a source of confusion.

Data Sources and Methods of Analysis

In this work, we draw on data taken from the final exams of 68 students enrolled in two introductory linear algebra classes at a large public university in the southwestern United States. Both sections were taught by the same instructor, who was a seasoned linear algebra instructor. Course topics included systems of linear equations, span and linear independence, linear transformations, determinants, eigenvectors, eigenvalues, and diagonalization.

	Four lines intersect in a point (Version L, part a, n=36)	Four lines intersect in a line (Version L, part b, n=37)	Four planes intersect in a point (Version P, part a, n=31)	Four planes intersect in a line (Version P, part b, n=31)
Correct systems approach	25% (n=9)	35% (n=13)	0%	0%
Correct rref & interpretation	35% (n=13)	24% (n=9)	35% (n=11)	42% (n=13)
Correct rref, not interpretation	11% (n=4)	11% (n=4)	48% (n=15)	42% (n=13)
Incorrect rref or other error	13.5% (n=5) 13.5% (n=5)	11% (n=4) 19% (n=7)	6% (n=2) 10% (n=3)	10% (n=3) 6% (n=2)
Total correct	60% (n=22)	59% (n=22)	35% (n=11)	42% (n=13)

Figure 2. Categorization of student responses

We examine student responses to four questions, two on each of two final exams. One final exam asked students to determine the intersection of four and then three equations of lines (Version L) and the other asked students to determine the intersection of four and then three equations of planes (Version P). Open-coding was initially used to categorize the solution strategies of students. Once codes were established, the two authors of this paper re-analyzed the data to come to consensus for each student on how their response should be coded. Major categories and the number of students in each category are presented in Figure 2.

Findings

In this section we illustrate a few sample results focused around (1) student use of multiple symbolic and graphic interpretations to correctly solve systems of equations of lines, and (2) student misinterpretation of the condensed (metonymic) form of the augmented matrix when not properly coordinated with other information given in the problem.

Solutions leveraging both systems of equations and augmented matrices

Students who solved the systems of equations that were lines were more likely to be correct: 60% and 59% for lines versus 35% and 42% for planes (see Figure 2). In addition they were much more likely to solve a systems of equations or use information from the system of equations to aide in their solution. Figure 3 illustrates examples of each of these approaches. In Figure 3a the student abandoned the augmented matrix to provide a detailed account of the solution by rewriting the equations of the lines. In Figure 3b, the student created a smaller augmented matrix based on their interpretation of the lines, and correctly row reduced it.

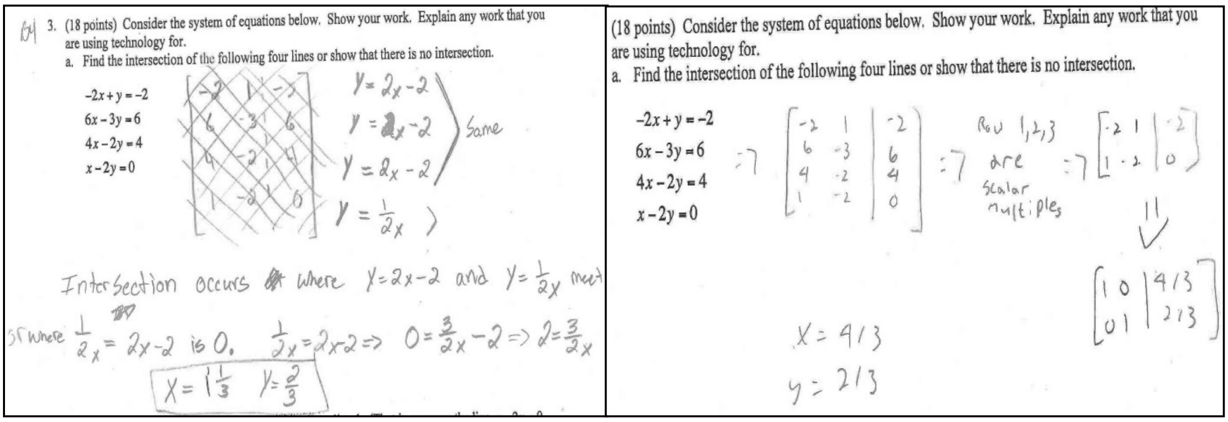


Figure 3a & 3b. Abandoned augmented matrix for a correct systems approach (version L)

The role of variable in using augmented matrices

When students implement a solution strategy using augmented matrices, the variables are temporarily removed from the equations for the purpose of row reduction. When the variables reappear in students' solutions, they sometimes reappear in strange ways. Of the 20 students who solved version P, part a incorrectly, 11 (55%) changed the variable names and/or the number of variables. Of the 18 students who solved version P, part b incorrectly, 10 (55%) changed the variable names and/or the number of variables. No students who correctly solved P and only one student who correctly solved version L renamed variables.

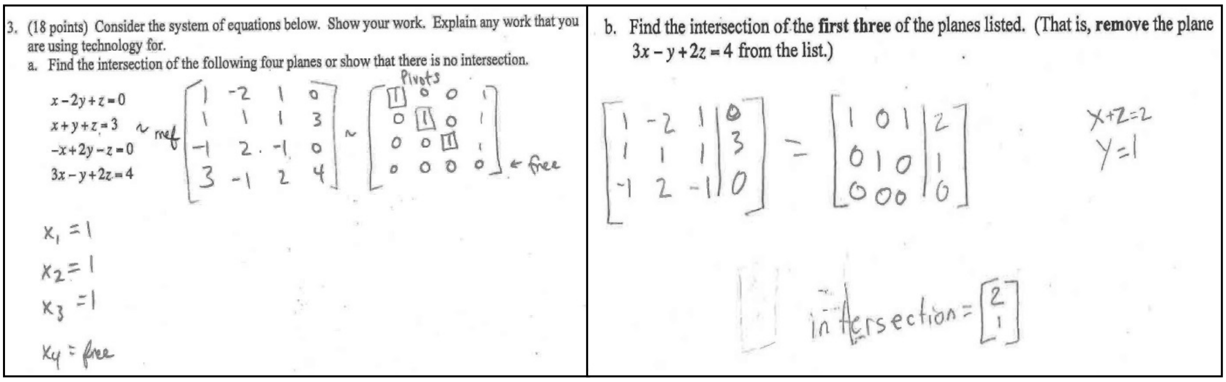


Figure 4. Correct row reduction (rref), but changed names and/or number of variables.

Curriculum Development

This brief report focuses on leveraging connections across different ways of interpreting systems of equations and their solutions as well as the challenges inherent in using condensed (metonymic) symbolizations such as augmented matrices. In our ongoing curriculum design work, we have developed and are testing new in-class activities to help students leverage their early work with vectors (Wawro et al., 2012) to better understand solutions to systems of equations (iola.math.vt.edu).

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9. TRANSITION: RESEARCH AND INNOVATIVE PRACTICE

Studying mathematics at university – Views of first year engineering students

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As part of a larger project investigating the transition from secondary to tertiary mathematics education in Sweden, interviews with 60 engineering students at the end of their first year of study suggest that shifts during the transition from individualised to collective study approaches and from dependent to apprenticed student positions, as well as student life, contribute to students' views on university studies as a generally highly valuable experience.

Introduction

A range of issues have been raised that relate to different types of problems students experience during their beginning undergraduate mathematics studies (e.g De Guzman, Hodgson, Robert & Villani, 1998; Gueudet, 2008), evidenced at many places in low participation and pass rates (e.g. Vollstedt, Heinze, Gojdka, & Rach, 2014). In their review, De Guzman, et al. (1998) pointed to epistemological and cognitive, sociological and cultural, and didactical issues connected to the *transition* from secondary to tertiary mathematics education. As observed by Gómez-Chacón et al. (2015), most studies exploring problems during this transition focus on cognitive aspects, such as the "abstraction shock" or cognitive parameters: "what we certainly can claim is that the success depends, in great measure, on the robustness of certain parameters in secondary education (attitude, motivation, approach towards work, and, in particular, learning styles and cognitive models) that might need to be significantly modified" (Clark & Lovric, 2009, p. 759). In their literature review, Bergsten and Jablonka (2015) show that the research on what sometimes has been termed the *transition problem* (e.g. Brandell, Hemmi, & Thunberg, 2008) addresses a wider range of critical issues: pass rates and participation; misalignment of curricula; changes in level of formalization and abstraction; unclear role of mathematics for the career path; differences in teaching and classroom organization; change in expected learning habits and study organization; differences in atmosphere and sense of belonging; and differences in pedagogic awareness of teachers. In this paper, focus will be on students' experiences of some issues related to the last four of these dimensions, using empirical data from a study of the transition problem in the context of engineering education in Sweden.

While in most studies the transition from secondary to tertiary mathematics education is framed as a problem, Hernandez-Martinez et al. (2011) see it "as a question of identity in which persons see themselves developing due to the distinct social and academic demands that the new institution poses" (p. 119), that is, as something potentially positive for future opportunities. Though conceived in different ways, the notion of *identity* has received an increased attention in mathematics education research, including the undergraduate level,

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and has been seen as decisive for capturing levels of participation in mathematical practices (Boaler & Greeno, 2000; Sfard & Prusak, 2005; Solomon, 2007). For the purpose of our study, we will here shortly discuss some recent research on identity in terms of student autonomy and individual learning strategies in relation to the institutional setting of university mathematics. In a study focusing on university mathematics students' perceived *autonomy* during first semester studies, Liebendörfer and Hochmuth (2015) conclude that autonomy depends on both the person, including competence, and the environment, and that learning strategies and institutional norms are critical. While "university expects students to work autonomously where students expect to be guided" (p. 9), the authors suggest that increased explicitness in demands on students may support their autonomy. Stadler et al. (2013) compared novice and experienced undergraduate mathematics students in Sweden and found that "beginners rely heavily on the teacher, while experienced students re-orient themselves from the teacher to other kinds of mathematical resources" (p. 2436), including their peers. The authors interpret this shift as an adaption to the "new learning environment". Similarly, Sikko and Pepin (2013) found, in a study conducted in Norway and the UK, that students learn more from collaboration with peers in tutorial or informal groups than from lectures.

Methodology

Sample and methods

The study presented here is part of a Swedish project (funded by *Vetenskapsrådet*) integrating the exploration of mathematical, didactical and social aspects of the transition from secondary to tertiary mathematics education. The project draws on data including different types of documents and interviews with engineering students and mathematics lecturers at two Swedish universities, including students' results from all mathematics exams during the first year of study. A total of 60 students, selected to represent different study programmes within engineering education as well as different mathematical achievement levels, were interviewed individually at three different points of time during their first year of study. In the third interview, conducted at the end of the year, students were asked to review their experiences from their first year mathematics studies and compare these to upper secondary school (high school). These audio-recorded semi-structured interviews lasted for around half an hour. The transcribed interviews were analyzed using a thematic approach (Bryman, 2004), with themes developing from students' responses to the prompts used in the interview guide. The following themes will be discussed in this paper: mathematics-related study habits at university and upper secondary school; differences between studying mathematics and other university subjects; expectations versus experiences of university mathematics; the role of mathematics teachers; the balance between study and student life; and the first year of study in retrospect.

Analytical framework

The analysis draws on the changes of student identities as learners of mathematics that are indicated by their responses to the interview prompts, organized as themes in our presentation of the findings. Discussing student/teacher identities in different schools, Dowling

(2009) observes that "students [...] were either *individualised* or organised *collectively* by teacher and/or student strategies" (p. 181). Teachers and students are in what Dowling (2009) describes as a "pedagogic relation", as opposed to an "exchange relation". The former can be recognised by the establishment of an author, an audience and a privileged "content", that is, a hegemonising practice/discourse aiming at closure, the evaluation principles of which are controlled by the author. Two distinct levels of subjectivity (or agency) attributed to "unauthorised positions" are "apprenticeship" (high level) and "dependency" (low level), which may amount to different hierarchical positions in relation to access to the principles of the hegemonising practice/discourse (Dowling, 2009, p. 244). We will use this distinction when discussing what our participants say about differences between being a high school student and a student at university.

Findings

The presentation of the findings will be organised by short summaries of the six themes mentioned above along with student quotes which, while being spread across engineering programmes and study result levels, are selected to be illustrative with respect to the themes. The findings will then be discussed in the final section of the paper.

Differences in mathematics-related study habits at university and upper secondary school

Most of the students report a change from a predominantly individual study approach at high school, with only little time spent outside lesson time, to a more collective approach working together with a group of peers after lectures at university or at home:

> *I study more with my peers than I have done earlier*

> *At high school I never studied with friends and not so much at home almost only at school, now I sit a lot with friends and at home, that's the difference*

Some students link university studies to doing things more on your own initiative:

> *Here you have to do everything on your own initiative, the teacher just presents it, you can ask but they are not present the same way*

> *At high school everybody is a little negative towards studying, here you have chosen yourself to do it, you have a more serious attitude*

Another comment concerns a difference in the separation of types of study activities:

> *Here it is more that you have a lecture and then there is lunch and then you do some math ... so it is more separated [than at high school]*

Several students report that over the year their study approach has become more focused on preparing the exam:

> *It has changed during the year, there is more cram for the exam now than in the beginning, then you did more exercises from the textbook and you followed the textbook carefully*

Differences between studying mathematics and other university subjects

Compared to other subjects in the engineering programme, studying mathematics is presented as different:

> *It is like night and day … the textbook in economy I read almost like fiction but the math books I rarely open … more like an encyclopedia*

> *It [mathematics] is very different from all the technical subjects*

The difference is sometimes expressed by the students in terms of coherence and their own understanding:

> *Math is quite smooth, other subjects can be pretty messy with broad facts that don't build up, do not require so much that you understand*

> *In math you must understand so math has been somewhat heavier than the other subjects*

Several students describe mathematics as more difficult than other subjects:

> *I focus mainly on math because math is harder*

> *High pace, more difficult to hang on, in math new things appear all the time until the exam, in other subjects you do a part and then you can let it go*

A common comment is that mathematics is more time consuming than the other subjects, for different reasons:

> *[Mathematics] takes a lot of your time maybe because you don't respect the other subjects as much yeah economics [laughter] you spend a lot more time on mathematics, it feels like it is more serious*

Expectations versus experiences of university mathematics

Regarding mathematics studies at university as compared to high school, it is commonly seen as more time consuming than expected and more demanding, both in terms of pace and effort:

> *The math studies are much more difficult than expected, much more to do … you just have to pass the math, it's not fun*

> *Maybe a little more than expected … one is almost fed up sometimes*

> *I was a little chocked the first period about how quickly it all ran … one week at high school is about two days here at most or even one day … it was very much higher pace*

> *Roughly [as expected] yes moves on a little quicker, you spend quite a lot of time for it*

There are many students who say they did not have any specific expectancies about studying at university, as pointed out also generally in research (Briggs, Clark, & Hall, 2012). Even then, however, the time issue is often raised:

> *Had no particular expectations, spending a lot of time on mathematics*

Some students who had expected mathematics to be rather difficult do confirm their expectations:

I heard that math is rather tough and had rather high expectations and that's where about it is

Had heard that many had problems with math, had tuned myself to that it might become tough and go for it from the start

The role of mathematics teachers

The university teachers are generally considered knowledgeable and helpful even if their pedagogical ability varies.

Knowledgeable … some teachers are able to explain so that difficult things become more easy to understand … if not one has to work more on it at home

They know more … some teachers are pretty bad some great

More helpful than expected

They do all they can to help you … their pedagogical ability varies

They are above expectations most of them I must say

That the contact with the university teachers is less personal seems to be less important as it is more efficient.

Maybe a little less personal now, a little more efficient here

Still the same [as at high school] willingness [to help] and it is just that one does not know each other that much

That university teachers are being less available than the high school teachers may be seen as negative, though this may be balanced by other advantages.

As expected … not so much contact with the teachers … not quite the same as at high school

Not as much contact like at high school but very nice, more on your own initiative, if you don't ask they will not explain

No demands from the teachers, they are here to help you, more whip at high school, more your own responsibility here

The balance between study and student life

Social aspects of life at university, which offers special opportunities ('student life'), are generally appreciated. While for some students, peer organized activities outside the studies risk to take over, others emphasise them as an important part of being successful:

I like it here, one of the reasons why it has run well, you are forced to learn to know more people to study at university, is among the best there is

Has been good, lots of fun, great socially, have managed it through this year, studied on and tried to have some fun

It's time to stop being that super social all the time and start living a more regular life

The first year of study in retrospect

Not many students, including those with low exam results, express thoughts of dropping out from their studies. However, some see mathematics as potentially contributing to such a decision:

> *Math is probably a big part of it if you drop out but it's not all*

Most students are generally positive about the first year mathematics studies, even if it sometimes has been a struggle:

> *It has been a good year, learnt a lot*
>
> *It has been like a roller coast [laughter] … it has been interesting and fun, did well so far*
>
> *I still think it is quite good that we have all of it [the mathematics] during the first year even if it has been tough*
>
> *guess it is a kind of needle's eye one has to go through somehow … it will be fine*
>
> *Can be nice to make your own decisions, this is quite good*
>
> *Incredibly rewarding to study at university, socially as well*

One student expresses a hope that the struggle he has experienced during the first year mathematics studies eventually will pay off:

> *What a feeling a mountain to climb … it's probably worth it because it will be good once you get out on the other side*

Discussion and conclusion

The outcomes indicate that studying mathematics stands out as different compared to other subjects regarding the large amount of time and effort that has to be invested, partly due to the specific character of mathematics courses that construct verticality in terms of generalization and specialization. It appeared to the students as more coherent (one student use the word "smooth") than other subjects; they framed this as a necessity to "understand" mathematics. In terms of identity, this can be interpreted as being *apprenticed* into the principles of the discourse. This they contrast to the situation at high school, where they did not need to study hard but only reproduce standard tasks, learned through exemplars to get a feel for the criteria for what counts as legitimate mathematics, without accessing the principles: they remain in a *dependent* unauthorised position. At university, as one student expressed it, there are "no demands from the teachers, they are here to help you, more whip at high school, more your own responsibility here". Students thus experience that they are expected to exhibit a increased degree of *autonomy* when starting to study at university (cf. Liebendörfer and Hochmuth, 2015). In a group interview conducted with some of the undergraduate mathematics teachers of these students (Bergsten & Jablonka, 2015), a similar interpretation of the pedagogic relation emerged: while at school the students are constructed as *dependent* learners, training a range of techniques using calculators and formularies with no authorship in original knowledge production, at university students are granted authorship to create mathematics at their level of competence by mathematical techniques and arguments acceptable by mathematicians. Such view of the need for more autonomy during the transition was also found in a study reported in De Guzman et al. (1998), as expressed by one lecturer: "they [the students] graduate high school feeling that learning must come down to them from their teachers. [...] That the students must also learn on their own, outside the classroom, is the main feature that distinguishes college from high school" (pp. 751-752). This need for autonomy was also implied by the way the students expressed the role of their teachers at university: although they were seen as being less available and "less personal" than at upper secondary school, teachers were described as very knowledgeable and there to help, based on the students' initiative.

Regarding study habits and learning strategies, for these students the transition to university mathematics constitutes a shift from an *individualised* to a *collective* approach, including informal group studies on their own initiative (cf. Sikko & Pepin, 2013; Stadler et al., 2013). They pointed out that individualised learning dominated at school. Despite the extra investment required for managing the mathematics courses in terms of time and effort, sometimes making the first year "like a roller-coaster" or "tough", most students found it "rewarding". One part of this appreciation seemed to be related to student life and that they were "forced" to get to know more people. Of these students, including all achievement levels, only a few appeared to have experienced the transition as a *problem*. To study at a university was by most students described as a generally highly valuable experience (cf. Hernandez-Martinez et al., 2011). Even though the transition might for some have constituted what one student termed "a kind of needle's eye", their appreciation of moving from a dependent to an apprenticed position appears to have contributed to this experience.

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Studifinder: Developing e-learning materials for the transition from secondary school to university

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For the platform Studifinder, we are developing interactive e-learning materials for mathematics in order to support students during the transition from secondary school to university. These interactive courses will be offered to prospective students from a wide range of disciplines and are designed to revise contents of school mathematics while introducing the notation and accuracy expected at university level. For each of the twelve topics a short and a long e-learning course have been developed. The material focuses on promoting a thorough understanding and also offers a wide range of exercises. In this paper we will present examples from these materials in order to illustrate our didactical design concept and describe how the course is embedded into the platform.

Introduction

In many fields of study mathematics plays an important role, but it is also a great challenge for beginners to handle. The students have to fill (often huge) gaps of their previous knowledge from secondary school to what universities expect them to know. Additionally, the transition from secondary school mathematics to university mathematics is difficult (c.f. de Guzman, Hodgson, Robert, Villani, 1998; Gueudet, G, 2008).

Therefore, most universities offer a bridging course to ease this transition and to prepare students for their course of study. Nowadays, bridging courses are often (but not exclusively) provided as online material because the students live far away from the university or have a job until they begin their studies.

To prepare all students independently from their chosen university the ministry for innovation, science and research as well as the universities and the colleges of higher education in North Rhine-Westphalia started a platform called Studifinder. It is an online platform where prospective students can get information about studying in North Rhine-Westphalia and also find an e-learning course called Studikurse for mathematics and the German language combined with knowledge and personal preference tests.

The Studifinder website

On the welcome page of the project Studifinder (http://www.studifinder.de), prospective students can get answers to four questions that they might ask when they want to take up studies. The first question is “Which field of study fits my skills and talents?” where the students can take the Studitest to find out what their aptitudes are.

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings.* Kassel, Germany: Universitätsbibliothek Kassel.
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The second question students can get answers to is "what courses of study are offered by which universities in North Rhine-Westphalia?" The Studisuche part of the website has an overview of which universities offers their favorite course of study.

If the students already know what and where they want to study they can take the Studicheck to test their skills in mathematics and the German language. After taking the Studicheck they get feedback in form of a traffic light. Each university can individually decide what percentage is necessary for the green or the yellow light. They can then take the Studikurs to repeat the relevant topics in mathematics and the German language to prepare themselves for their studies at the university. Each Studikurs is tailored to the results of the Studicheck.

The Studikurs and its constituent parts

Our working group is responsible for the development of the mathematics Studikurs for the Studifinder platform. The new learning material is based on the VEMINT course that has been used by several universities for over a decade to offer a blended-learning bridging course in mathematics (c.f. Bausch et al, 2014).

One requirement was that the Studikurs had to harmonize with the Studichecks, which have been developed by a team at the RWTH Aachen, as well as the standards of education in mathematics for the secondary schools in North Rhine-Westphalia (c.f. standards of education North Rhine-Westphalia 2012) and the COSH-Catalogue (c.f. COSH-Catalogue 2014) to define which topics have to be covered in the material. The material provides information on twelve different topics, namely:

Released in September 2015	**Expected release in March 2016**	**Expected release in August 2016**
Basic functions	Linear systems of equations	Geometry
Power, roots and logarithm	Basic arithmetic	Trigonometry
Terms and equations	Integral calculus	Vectors and analytic geometry
Differential calculus	Higher functions	Stochastic

Table 1. Overview of the courses and the related release dates

We develop a long course and a short course for each topic to satisfy two major needs. The short course is designed to give a quick overview of a topic and the long course is designed to convey detailed information about the topic. It takes about 45 to 90 minutes in a short course and about six hours in a long course to work through the material.

Outline of a short course

In the short course the topic is motivated with the help of an introductory exercise often from another field of study, for instance physics or economics, to demonstrate where the mathematical content can be applied. It also provides short and concise information about the topic to get an overview of what is required to solve the exercise. Within the introductory exercise, it is possible to stop at certain points and lookup the mathematical background via cross-references. In order to easily navigate through the material, tabs are supplied on top of the content area to switch between the introductory exercise and the mathematical input. At the end of a mathematical input we place a button to jump back to the introductory exercise.

The short course also offers an opportunity for the students to see if they can work with online-materials successfully. The short course includes (besides the interactive exercises) applets, animations and short videos to motivate the students to work through. At the end of a short course there is a preview about what they can expect from the respective long course.

Outline of a long course

In the long course the topics are explained in detail on a level between school mathematics and university mathematics. In this part of the course, we emphasize the importance of correct notation and accuracy in mathematics as will be expected at university. So the students can repeat their school knowledge from an elevated perspective on the one hand and gain a thorough understanding of the topic on the other.

Therefore, the long course is divided in up to six separate subchapters which are built-up step by step. The subchapters are once more divided into an overview, introduction, explanations, exercises, applications and (optional) supplements. The students are motivated to learn with the interactive exercises and Geogebra applets in the material. We typically use animations and videos in the long course to explain some issues. You can find two examples in fig. 1. Throughout the contents, interactive elements like videos are offered to create possibilities for varied learning paces and hence keep up the learning motivation in the long run.

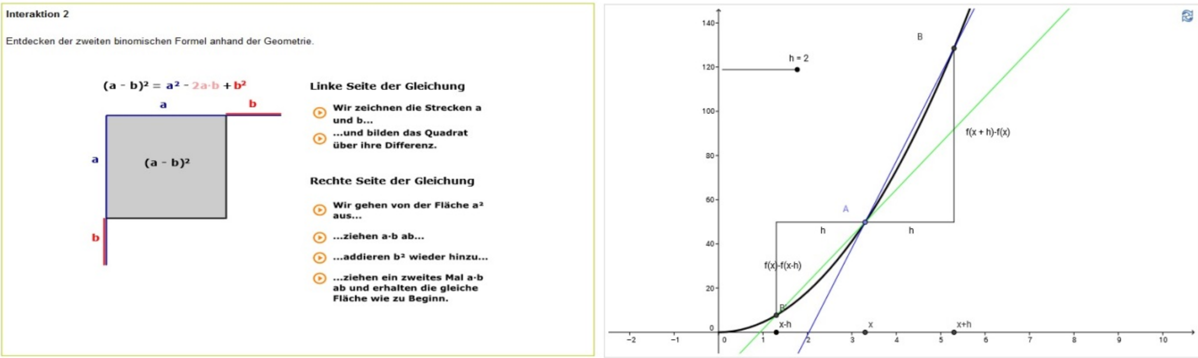

Fig. 1. On the left a step by step interactive explanation of how to square a binomial. On the right a Geogebra applet to visualize the principle of differentiation.

Each course includes a glossary of mathematical symbols and an introduction of how to use the formula inputs in the interactive material correctly.

With our material the students can decide which topics and how fast they want to learn. This creates an environment of self-guided learning. Especially for designing an effectively individual learning path in self-regulated learning, it is important to know the individual strengths and weaknesses (c.f. Niegemann et al., 2009, pp. 295).

Visualizations and interactive feedback

To create an interactive user experience for the learners we put some effort into the arrangement of the material. Every exercise in the learning material comes with a detailed solution. Learners can access them by clicking the button "show solution" and hide them again at will. Once evoked, the solution places itself naturally in the flow of the contents, so it can be on the screen together with the original exercise. Furthermore students can use automatic feedback to check their own solution before they take a look at the right answer. This gives them the chance to reflect on their own solution. Since "feedback students receive from the teacher is a key supportive factor of the process of continuous improvement" (Ibabe & Jauregizar, 2010, p. 244) this provides more objective feedback than pure self-evaluation. Because entering mathematical terms into a system on a keyboard is not easy, we designed an on-the-fly formula interpretation (the VE&MINT project developed a similar technique based on the mparser), which shows the learner the interpretation of his/her input before the answer is evaluated.

Further Work

The first four course packages were developed and made accessible in September 2015. That material has been evaluated and will be evaluated every time we submit one of the packages. A working group in Aachen led by Miss Wachtel is responsible for the evaluation. The last course packages will go live in August 2016. As soon as the development stage is complete we will start a quality management period until December 2017, when the project is planned to end.

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Didactic contract and secondary-tertiary transition: a focus on resources and their use

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Abstract: In this article we claim that the concept of didactic contract can help to develop a deeper understanding of the secondary-tertiary transition, in particular by showing changes at three different levels: at the general, institutional level; at the level of the discipline concerning mathematical practices; and at the level of a given mathematical content. In fact, we argue that the didactic contract is linked to the use of and interaction with different resources, by teachers and students, in the sense that their use is shaped by the contract; and at the same time the available resources shape the mathematics taught. We draw here on two studies, one in the UK and one in France, to illustrate how a focus on resources can inform us about contract rules at the different levels.

Didactic contract and interaction with resources: framework and research questions

The study presented here is a contribution to research on the transition from secondary school to university mathematics (Gueudet 2008; Pepin 2014). Whilst different theoretical perspectives can enlighten what happens during this transition (Nardi et al. 2014), we retain a socio-cultural approach in this paper. We consider secondary school and university as two different institutions (Chevallard 2006), with different mathematical practices. In particular, the didactic contract (Brousseau 1997) is different in these two institutions. The didactic contract is defined as a set of rules, some explicit but most of them implicit, framing the mathematical practices of both teacher and student/s, which can be presented as a sharing of responsibilities between teacher and student/s. Moreover, we have a specific interest in the links between the didactic contract and the resources intervening in the students' mathematical work. In previous works (e.g. Gueudet, Pepin & Trouche 2012) we have shown that the use of resources – we consider here curriculum resources, like textbooks, websites, but also lecture notes, for example – contributes to shaping mathematics instruction and learning; and it is likely to shape in particular the didactic contract.

The didactic contract can be considered at different levels (Chevallard 2006, Winsløw et al. 2014); we distinguish here between three such levels (De Vleeschouwer & Gueudet 2011):

- a general, institutional contract: its rules apply for all subjects taught in the given institution. For example, at secondary school the teacher writes on the blackboard all that the students need to write down; at University, at least a part of the content is only provided orally by the teachers and the student him/herself is responsible for taking written notes. We interpret this as a change of contract rules during the transition, likely to raise difficulties for many students.

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- a contract at the level of the subject (here, mathematics): its rules apply for all mathematical contents. For example, new expectations at university in terms of rigor are changes in the didactic contract at this level.
- a didactic contract for particular mathematical contents: here the rule concerns a particular mathematical content, like Linear Algebra, Calculus and so on. Such rules are likely to change during transition for contents taught both at secondary school and at university (e.g. different teaching of calculus: "calculational" at school (how to calculate an integral); and more mathematical at university (precise definitions and assumptions, etc.)).

Our central research questions are the following:

What are the resources and what is their "expected" (by the institution) use by students in higher education mathematics (first year), as compared to the resources and their uses at secondary school? How do these resources and their uses inform us about the didactic contract at each level?

We draw on two different data sets: one in the UK concerning the student transition from upper secondary school to university mathematics[1]; and one in France concerning the teaching of number theory at first year university. To emphasize, we do not intend to compare these two cases; we consider them as complementary, since the first one informs us on the contract at the general and at the subject level, while the second informs us at the subject and the content level.

Resources and institutional contract: the Transmath project

In the UK, our data are anchored in the TransMaths project, where we investigated how students experienced different mathematics teaching-and-learning practices at both sides of the transition point, and they developed different strategies to make the transition successful (or not). As this project used a mixed-method approach, we could identify and analyse particular resources and their use, both from interviews with students (and lecturers), and from the accompanying student survey.

From interviews with Sunny (and his friends), who studied at City University, we could identify the main resources used in their first year of study (see appendix): lecture and lecture notes; the lecturer him/herself (during office hours); the coursework, tutorial and tutor; their friends/study group. It was clear that these resources were quite different, in nature, from what students were used to at school: at school students had a textbook (which was portraying mathematics as something that one can learn by solving "tons of exercises"); and the teacher who was available for individual questions and explanations during lesson time (and even out of school time for special revision lessons). Friendship group did not seem to be important, as students could discuss their problems with the teacher, who was seen as the authority in terms of correctness and learning of the subject.

[1] Transmaths project at the University of Manchester: http://www.transmaths.org

Indeed, at university one mathematics lecturer said that "students [have to] learn 'from day one' that they are not in school but in a university mathematics department". This included a clear distance between students and lecturers, which was also mentioned by Sunny:

> *"I think it's, it's more like the learning here is more general like in a way, like in sixth form it was more personalised kind of. You kind of, you was closer to the teacher, you was, you had constant like, you was talking to them – you was after school you was chatting to them. You saw them around, like here it's so funny cos when we see the lecturers walking around it's like they're like celebrities. Cos we haven't got, we haven't quite got that personalised you know, thing with them so they're from a distance you know. 'That's Professor …, that's Professor…wow!' You're like wow, they're about. So I suppose it's less personal in a way." (DP5, Sunny)*

Returning to resources used and their nature, at university the main resources were clearly the lecture and the lecture notes provided by the lecturer/professor (sometimes supported by a textbook). However, these were not always "understandable" and students would have needed individual help from the lecturer (during particular office hours), but most students did not dare to go (as they were afraid of asking "stupid questions"). In addition, the coursework (provided once a week) was to support student understanding of the lecture, through exercises. Sunny and his friends/learning group emphasized that unless the coursework was well aligned with the lectures, it did not help their understanding of the topic (see calculus as compared to geometry lectures/coursework). Indeed, in some cases students did not know what to ask in tutorial time, or in lectures, so little had they understood the topic area. Other resources included textbooks (suggested/approved by the lecturers); and particular self-support schemes (where higher year students help their 'younger' peers) – these were seen as less helpful than the notes and support provided by lecturers and tutors, in particular as students were often "learning to the test". However, the same resources (e.g. lecture notes) were often evaluated very differently by students, in terms of support for their learning, so much so that Sunny (as student representative) had asked for a change in form and practice concerning lecture notes: as students did not want to be presented with "one slide after another", they asked for hand-written lectures during the lecture (so that they would have time to think and process the notes, and perhaps ask questions).

At the same time institutional practices and accompanying resources played a crucial role in the ways that mathematics, and what it meant to "do mathematics", was portrayed, which often hindered students developing a mathematical disposition that supported their engagement with demanding mathematics. From the student surveys at entry to university, as compared to a year later, we could see that students adjusted to particular practices and routines, and socio-mathematical norms (see Pepin 2014). In particular, "whole-group/class teaching" (and listening/writing in lectures) and "working in groups" (either with friends, or in the tutorial) was seen as essential to pass the examinations. "Taking notes in lectures" and "studying from your own notes" depended on what the lecturer provided as learning resources. For example, one lecturer (of geometry) apparently provided "perfect notes", that is lecture notes that suited students' level of understanding and learning pace, and that were aligned with the coursework (exercises) and the examinations. So, students felt well-prepared by the lectures, the lecturer's explanations during the lecture, and the coursework, to pass the examinations.

On the basis of video footage of selected lectures and pre- and post-video stimulated recall discussions with lecturers, one could identify meanings that were attached to particular practices. Particular lectures reflected the kinds of things that a “rigorous mathematician” may need to learn:

- ‘reasoning and proof’ based thinking and practices were expected to be developed through Geometry and Linear Algebra;
- ‘procedural fluency’ (methods) was seen to be developed through Calculus;
- practical and context relatedness was regarded to be developed through Statistics.

In terms of Didactic Contract for mathematics, it can be argued that there was a clear *institutional didactic contract* at City University, made explicit in discussion with lecturers and students, and mediated by particular practices. This contract was about helping students to become a “rigorous mathematician” and attain the “very highest academic standards”. Becoming a “rigorous mathematician” included making sense of the mathematics in lectures, and different lectures (different mathematical topic areas) appeared to provide the key to particular competencies (e.g. reasoning and proof was developed through Algebra). However, how students were expected to learn and develop these was not clear. It can be argued that this change of Didactic Contract from school to university appeared to necessitate students becoming more independent learners. In terms of resources (and their use) we retain that the change of Didactic Contract at transition from school to university mathematics education has implications for students, (1) in terms of the change in nature of the resources: teachers ‘change into’ lecturers; lessons into lectures; homework into coursework; textbooks into course materials and lecture notes; tests into examinations; and school mathematics into university mathematics; and (2) in terms of the expectations of their use: the teacher could be accesses (nearly) all the time, whereas the (individual) lecturer is ‘only’ available for a limited number of minutes/hours; textbooks in school are seen as a support of teachers’ teaching, mainly in terms of provision of exercises, whereas at university lecture notes are to be “understood” and studied, with the support of the lecture and the coursework/tutorials.

Resources, didactic contract and mathematical content: the case of number theory

In this section we focus on the Didactical Contract at the level of a particular mathematical content. Number theory is known as a difficult topic for students of different levels (Zazkis & Campbell 2006). In France, where our study took place, the scientific students taking the “mathematics specialty” in grade 12 (last year before university) learned advanced topics like prime numbers, Bézout’s and Gauss’ theorems and congruencies, with the aim to develop reasoning and proof skills. However, Battie (2010) who examined exercises proposed at the Baccalauréat (end of secondary school examination) argues that the expectations for grade 12 students concerning number theory was mostly limited to computation, and the application of methods they learned. We interpret this in terms of the Didactic Contract (Brousseau 1997): an important rule of the Didactic Contract for number theory in grade 12 was that developing an original solution method was not part of the students' responsibilities.

At university level we investigated a teaching unit on number theory taught at the first semester of the first year in a university in France. This teaching unit addressed topics such as: Euclidean division and Euclidean algorithm; prime numbers; and congruencies. The main resources available for the students in this teaching were actually the mathematical "texts" (e.g. exercises from textbooks; etc.). We firstly consider the level of mathematics as a discipline.

We asked the students about their use of mathematical texts (considered as resources) for this teaching unit; we also asked the responsible lecturer of this course about the uses she would expect (from students). We proposed an online questionnaire to the 140 students enrolled and obtained 85 answers. The resources offered by the institution were: a "polycopie" (lecture notes, comprising of all the definitions, the theorems and their proofs – instead of a textbook); exercise sheets; previous examination papers, all on paper and available online as pdf files. Moreover, the students had their own course notes (five classes followed this course, with five different teachers). According to the lecturer responsible for the course, the students should work on the polycopie in order to learn the theorems and to work on the proofs. Considering the answers to our questionnaire, the actual situation seemed quite different: only 52% of the students declared that they found the polycopie helpful. They considered that the lecture was enough, and that they used the polycopie only for the final examinations (83%). Moreover 90% would like to find worked examples in the polycopie; and 44% looked for additional resources on the Internet, in particular worked examples.

We interpret these answers as follows: whilst the teachers at university expected that the polycopie would be used to work on the text of the lecture (i.e. in terms of definitions, theorems, proofs etc.), the students considered that their responsibility was to work on the exercises, and that this was an efficient way to prepare the test. In secondary school they were used to find methods presented in the textbook, in particular many worked examples (Rezat 2013). In this teaching unit, the polycopie did not incorporate the presentation of methods how to solve problems, or worked examples. In fact, most of the exercises proposed did not correspond to the application of a given method.

Let us now consider the level of a particular mathematical content. Differences between the school and university Didactic Contracts could be observed by analyzing the questions/exercises proposed (e.g. in examinations, or in tutorials). For example, at the final examination the following exercise was proposed (figure 1):

Exercice 4 *3 points*

On définit une application de $\mathbb{N}$ dans $\mathbb{N}$ par $f(n) = \text{pgcd}(42, n)$ pour tout entier naturel n.

1. Calculer $f(0)$, $f(2)$, $f(10)$ et $f(5)$. L'application f est-elle injective ?

2. L'application f est-elle surjective ? Déterminer $f(\mathbb{N})$.

Figure 1. Exercise 4. An application from IN to IN is defined by f(n)=gcd(n,42) for all integer n. 1. Compute f(0), f(2), f(10) and f(5). Is f an injective application? 2. Is f surjective? Determine

***f(IN)*. The exam lasts two hours, and comprises of 6 exercises. The exam is marked on 20 points, exercise 4 is marked on 3 points.**

In this exercise, for the first questions the students just needed to apply the definition of a gcd to compute $f(0)$, $f(2)$, $f(10)$ and $f(5)$. Then they found that $f(2) = 2 = f(10)$, and concluded by applying the definition of injectivity that f is not injective. The second question required more personal initiative. The students had to determine the range of f; the question is "open", thus they firstly need to decide if they will try to prove that f is surjective or not. Then they searched for $f(IN)$. This required to choose the relevant property of the gcd: gcd(n, 42) divides 42, so it belongs to $\{1, 2, 3, 6, 7, 14, 21, 42\}$; and then to justify that each of these values was reached by f.

No similar exercise had been solved in the tutorials, or given as example in the polycopie. Surjective functions was indeed a topic studied at the beginning of the teaching unit, during the second or third week, while gcd was studied during the weeks 8 and 9. In the tutorials, no exercises associated gcd and functions.

Analyzing the mathematical problems/exercises proposed (in tutorials, examinations or in tutorials) showed that students were not given "recipe" solutions, but that they were expected to use their understandings of the course lectures to find a solution method. The example given above with gcd(42, n) was typical: the students had to go back to the definitions and properties presented in the course (here: the gcd of two numbers is in particular a divisor of these two numbers) to build their own solution method. Thus, unlike secondary school, in the Didactic Contract at the level of mathematics at university, building the method was the students' responsibility, and this would direct their use of resources for their individual work. However, in the first year at university the students had not yet entered into this contract, they still looked for worked examples in order to observe and reproduce solution methods.

Discussion

Our investigations led us to observe changes in the rules of the Didactic Contract between school and university, at the institutional level and at the level of mathematics in UK; at the level of mathematics and at the level of a particular content in France. These rules were associated with the use of particular resources, which subsequently became indicators for these changes.

At the institutional level, we retain from our study in UK the increasing responsibility of the students towards their own developing understanding, and the "replacement" of the teacher by other students as a central resource. At the level of mathematics, both in UK and in France we observed that the lecturer expected that the text of the lecture would be used by the students, not only to learn and understand the concepts, but also as a model for his/her own mathematical practices, mathematical proof in particular. In France we observed that the novice students did not adhere to this rule, and searched for worked examples as model. At the level of a particular content, number theory, we observed expected changes in the students' responsibilities through and by analyzing the mathematical texts. At university this also included developing or identifying/choosing a method to solve an exercise.

We claim that the available and expected usages of resources contributed to the shaping of the Didactic Contract. At the same time the resources were shaped by the teachers' and students' expectations. Hence, we contend that investigating the changes in resources and their actual or expected usages can inform about changes in the contract, at different levels.

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Integrated course and teaching concepts at the MINT-Kolleg Baden-Württemberg

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The MINT-Kolleg Baden-Württemberg, a joint institution of the Karlsruher Institut für Technologie (KIT) and the University of Stuttgart, offers a wide range of supplementary courses at the transition level, both for interested candidates and already enrolled students. An integrated course concept is used to bridge the gap between school and university in mathematics, computer science, physics and chemistry. Course types range from classical lectures to small group teaching, inverted classroom concepts and online learning and testing. The need for different course types arises from a complicated mixture of reasons for problems in the first year at university. Combination of different course types proves to be more successful in raising the success ratio then a single course type program for specific audiences.

Introduction

Transition from school mathematics to university level is one of the most difficult steps for a beginning student due to a multitude of reasons. Difficulties mainly arise from new paradigms of thinking about mathematics as described in [Grünwald], from a widening gap of content taught in school to content assumed to be taught at school by university lecturers, from diverse and changing ways to acquire an university entrance examination (Hochschulzugangsberechtigung in German), but also from miscellaneous problems like finding accommodation at the start of the term, being away from family for the first time and having a much tighter time and work schedule then before. If this transition and the necessary change in the learning and working behavior occurs too late (or not at all), an effective understanding of lecture content becomes impossible. To address these problems, a wide range of supplementary course types is offered by the MINT-Kolleg, differing by teaching strategy and setting (lecture, guided exercise solving, online learning, group based working, open learning rooms), time schedule (before enrollment, before term, parallel to the lecture, after term, prior to examination) and subject (courses of MINT type: mathematics, computer science, natural sciences and engineering, but also interdisciplinary courses like time management and efficient learning techniques). Also each course type comes with its own testing and feedback tools. Students combine courses of different subjects and types to improve their examination results, adjusted to personal preference or time constraints.

Course types at the MINT-Kolleg Baden-Württemberg (KIT)

Preparatory courses (studienvorbereitende Kurse in German) are offered for candidates not yet enrolled at university. Applicants select and combine courses in mathematics, computer science, physics or chemistry. Courses are taught in small groups of up to 20 students. Content is selected both from school curricula and university level courses to both repeat necessary school topics and introduce university level concepts to deal with these topics. These courses are advertised for prospective students having a larger time gap between school and university enrollment. Typically few students enlist for these courses, mostly those planning to start a bachelor program after an apprenticeship or prolonged military or civil service.

Bridging courses (Vorkurse in German) are offered for enrolled students one month (typically in September) before regular courses start. They consist of a classical 90 minute lecture with up to 200 students in each course followed by small exercise solving groups (Tutorien in German, up to 25 students in each group) led by student tutors. Depending on the subject and its target bachelor program these courses often exceed school level, especially in mathematics where students are required to perform complicated calculations without using a calculator or a computer algebra system. Typically, 20% of the students enlist for bridging courses and combine different subjects, for example mathematics and computer science.

Online courses and tests are offered during the whole year for free, some are integrated into certain bridging courses. Focus is strictly on repetition or assessment of school level content.

Accompanying courses (Begleitkurse in German) take place during a term. Most calculus and linear algebra courses at the KIT consist of a classical lecture, an exercise lecture (Große Übung in German) and small exercise solving groups (Tutorien) totalling 4+2+2=8 hours. An additional accompanying course (totaling 6 hours) is offered for students who are unable to follow the lecture or do not understand how to solve exercises using the contents of the lecture. These courses are taught in small groups by professional teachers (which usually hold a PhD and have several years of teaching experience). Content given in the lecture is briefly repeated in the course and then intensely trained by solving guided exercises. Typically, 10% of the students enlist for such a course.

Basic courses (Basiskurse in German) repeating basic mathematical skills are available during the term for students who show severe deficits in school mathematics. These courses are not connected to a specific lecture and students from different Bachelor programs participate in the same course. An average of 2% of students attends a basic course in mathematics.

Countercyclic courses (Antizyklische Kurse in German) are offered for students who failed to pass an exam and typically repeat a specific lecture. Like accompanying courses they consist mostly of guided exercise solving, as content from the previous lecture is known but not fully understood. Typically, 5% of the students opt to enlist for such a course.

Repetition courses (Aufbaukurse in German) are offered prior to a written examination between terms. They intensely repeat and train techniques and content from a lecture. These are offered as compact courses (four hours each day for up to two weeks), typically by the same teacher who gave the accompanying course. Typically, 10% of the students enlist for repetition courses.

Interdisciplinary courses are offered like accompanying courses alongside lectures. They compromise time scheduling, working and learning techniques, support courses for women or language courses for migrants.

Open learning rooms (Offene Lernräume/Helpdesks in German) are offered at fixed weekly time slots in mathematics and computer science. Participants are encouraged to form small groups to solve weekly homework (Übungsbetrieb in German) or prepare for an examination. Professional staff is present to help and render advice, or briefly repeat lecture content, but group based problem solving by students themselves is the primary focus of this concept.

The preferred combination at the KIT is a bridging course (both in mathematics and computer science) and an accompanying course for the first mathematics lecture (calculus in most cases). An average of 10% of first year students in MINT bachelor programs at the KIT settle for this combination. A detailed analysis of this specific combination and its effectiveness is described in [Ebner et al]. Students failing their tests in accompanying courses are advised to change to a basic course in mathematics and can do so without further requirements. Often students start attending accompanying courses in the middle of the term, when lectures become more complex and they are no longer able to solve their assignments by themselves.

Teaching, testing and course integration

Teaching in most course types at the MINT-Kolleg is an interactive process, often course structure and content selection is changed according to test and supervision results, sometimes students are advised to change from an accompanying course to a basic course. In extreme cases especially difficult topics are explained in full detail while other content from the lecture is omitted. The following testing concepts are applied by the MINT-Kolleg:

- Bridging courses use tests both at the beginning and the end of the course to assess both the level of knowledge of beginning students at the start of the term and the success of the course itself.
- Preparatory, accompanying and countercyclic courses use introductory tests and short midterm tests, but students are judged mainly by their (supervised) exercise work during course times, giving teachers an immediate and precise feedback on the success of their teaching style.
- Online learning and testing information is gathered also, but cannot be related to participants during the term because online testing at the MINT-Kolleg is mostly anonymous.

Test results show a consistent composition of the audience in MINT courses consisting of students who want to refine their skills (but not in need of the course itself), students who are unable to understand lecture content sufficiently to solve exercises, students who understand the lecture but suffer from a severe lack of mathematics basics taught at school (the typical example being a student who is able to compute the characteristic polynomial of a matrix but unable to factorize it) and students who have lost track of the lecture and its exercises entirely. These groups are not restricted to a lecture: students having problems in a math lecture often have the same problems in physics or computer science lectures and typically combine accompanying courses for both. In extreme cases students attend accompanying courses for every mathematics and computer science lecture until they become comfortable at the university level and drop courses they no longer need.

Results

While test data (both from MINT courses and final lecture examinations) confirms a rise both in grades obtained and percentage of passed tests for participants of MINT courses with respect to students attending the regular lecture program only, strong improvements are found for students who have attended a combination of MINT courses. For example, combining a bridging course and an accompanying course proves to be much more successful then single course selections, as is shown in [Ebner et al]. Even unusual or excessive combinations (for example students attending accompanying and countercyclic courses simultaneously, adding up to 12 hours to their regular workload) succeed and result in significantly higher success rates. These results are not related to mathematics and are found in computer science, physics and chemistry also.

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Project *mamdim* – Learning mathematics with digital media

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The BMBF-project mamdim explores novice students' mathematical competencies as well as the usage and benefit of digital media focusing on descriptive statistics. In the main study next year, 300 students from five participating German universities will work mainly on measures of center and spread using media such as video tutorials, audio commented presentations and instructional texts. There are different study settings regarding the number of students (alone or in dyads) and the task (with respectively without accompanying questions/prompts). First results from a pilot study that was conducted at two universities (N = 68) will be presented and implications for the main study drawn.

Introduction

The transition from school to university is well known as problematic. Especially the transition in mathematics is explored by several researchers (e.g. Artigue 2001; de Guzmán et al. 1998; Wood 2001) and described as a complex problem area consisting of individual, social, epistemological, cultural and didactical impacts (de Guzmán et al. 1998; Gueudet 2008). Since decades universities try to support their students in the first semester by offering bridging courses to get them accustomed to the way in which mathematics is taught at university level. An analysis shows that the impact and use of digital media has been growing during the last years (Biehler et al. 2014 I), whereas the impact of these learning environments on the learning process and the effectiveness of learning is nearly unexplored. In these learning environments respectively bridging courses digital media such as pdf-documents, interactive pdf-documents, screencasts, videos, online-tests, animated worked-out examples, dynamic-geometry-environments (DGE) and computer-algebra-systems is used. Moreover, a comparison of several digital learning environments does unfortunately not happen in current research (Biehler et al. 2014 II). The project *mamdim* takes these learning processes into its research focus by exploring students' handling of different digital media from several perspectives.

The project *mamdim*

The project *mamdim* (mathematics learning with digital media in the passage from secondary to tertiary education) is financially supported by the German Federal Ministry of Education and Research (BMBF). The main study will take place in summer 2016 in bridging courses at the German universities of Bielefeld, Cottbus-Senftenberg, Pforzheim, Oldenburg and Offenburg. Every university uses a specific digital medium in the bridging course and in every course the topic measures of center and spread in descriptive statistics is picked out as a central theme. In this paper, the focus is on the pilot study which was conducted at the universities of Bielefeld and Offenburg in autumn 2015 with 68 probands in all.

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings.* Kassel, Germany: Universitätsbibliothek Kassel.
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Aims and Methodology

The research questions are the following:

1. What kind of communication processes take place, when students work in dyads on the digital medium and how can these communication processes be stimulated?
2. What influence do these communication processes have on the process of learning mathematics and how do these learning processes differ from that of a single user of the medium?
3. How effective is the use of a specific digital medium concerning the learning effect of individuals with a comparably similar knowledge before the intervention?
4. What influence does the use of digital media have on students' motivation?
5. Is the digital medium utilized by students in the expected manner and which learning difficulties can be observed?
6. Is it possible to identify different user types of digital media?

To answer those research questions, the following design was developed for the main study at five universities each with 60 probands (fig. 1):

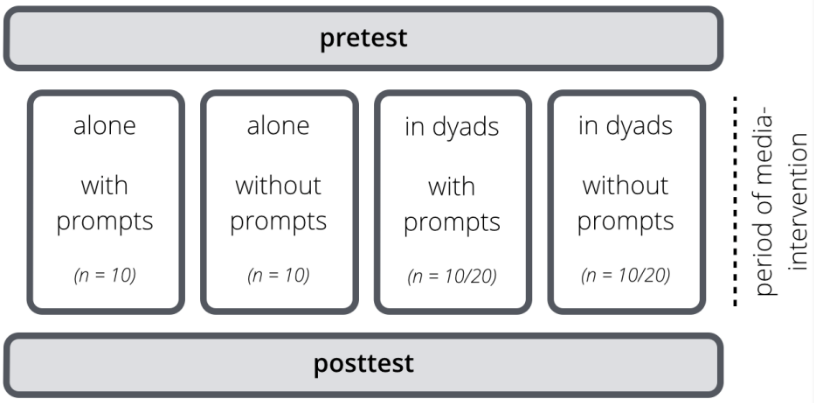

Figure 1: Study design

The students' knowledge regarding descriptive statistics is tested before and after the media-intervention-phase with the help of paper-and-pencil-questionnaires. Furthermore, their (academic) motivation, their domain-specific self-efficacy and the acceptance of the material used during the intervention period are surveyed. Moreover, the user behavior and their learning strategies during this phase will be analyzed qualitatively using the video recordings from the interventions as well as the students' individual notes.

First Results and Implications for the Main Study

The main aim of the pilot study was the improvement and evaluation of several instruments with regard to the main study. The first version of the pretest, constructed for the assessment of the students' prior knowledge, consisted of 21 items. Due to the mathematical focus of the study, the items deal with the calculation of measures of central tendency and variability as well as with their application in real life situations. First analyses reveal that many items show solution frequencies from 0% up to 40% and only a few items are solved with frequencies higher than 50% (fig. 2). To improve the test balance, a selection of difficult items (solution frequency below 10%) shall be replaced by easier items for the main study.

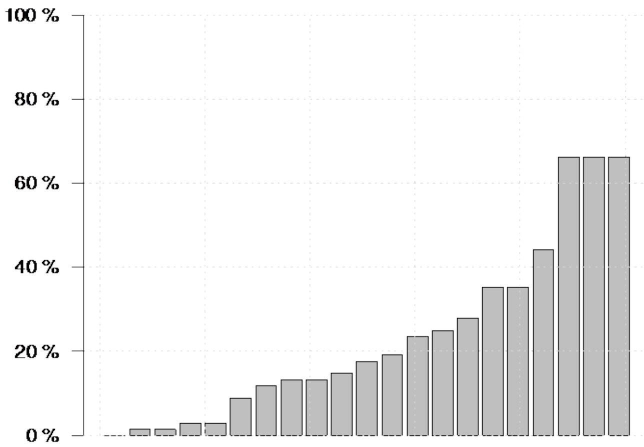

Figure 2: Solution frequencies of pretest items

Regarding the acceptance of the material in the pilot study, the instructional video format was highly accepted by most probands with 91% agreeing it was fun to work with it and a good help for learning (97.1%). Students emphasized the importance of many comprehensible, realistic examples, an appropriate length of the video respectively the verbally annotated presentation and their preference for controlling the learning speed e.g. by individually rewinding or fast-forwarding the digital medium. It will be interesting to see through further analysis especially of the video recordings taken during the intervention phase, in how far the different settings (alone – in dyads/with or without questions or prompts) influence the communication processes and hence, the learning outcomes.

Outlook

With the results of the ongoing analyses of the empirical data collected during the pilot study (video recordings, motivation scales and tests/questionnaires), we will improve the quality of the various instruments with regard to the main study.

Furthermore, some aspects of the instructional videos and presentations will be adjusted to the students' feedback. With regard to instructional design guidelines, influences of suboptimal design features of the material shall be minimized, having in mind the different basic conceptions, aims and peculiarities of the partner universities.

These modifications will be completed in spring 2016, so that the main study with a total of approximately 300 students from different courses at the partner universities can be conducted from March 2016 on.

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Design, conception and realization of an interactive manual for e-learning materials in a mathematical domain

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Multimedia learning materials increase in popularity – especially in the context of bridging courses, which aim to ease the students' transition from school mathematics to university mathematics. At the same time they become more and more sophisticated in their design on the level of contents as well as in the didactical environment they provide. We will discuss how a high level of support for self-regulation in multimedia learning materials makes a learning environment less self-evident and increases the need of an improved guidance for learners. Afterwards we will present an interactive manual that was designed to efficiently support self-regulated learning in VEMINT's e-learning materials for bridging courses further and its underlying design principles.

A (very) brief overview of VEMINT and its learning materials

VEMINT is a German acronym which loosely translates into virtual entrance tutorial for STEM. The VEMINT-project is a collaborative project by the universities of Darmstadt, Kassel, Lüneburg and Paderborn with members coming from the field of mathematics and from mathematics education. VEMINT's learning materials are multimedia based e-learning materials. One important characteristic is the structure of the contents, which are organized into 60 different modules. They each present a domain (e.g. linear functions) in a closed form and independently from other modules. This organization of contents allows a unique selection of modules for many different (preparatory) courses. Furthermore each module is structured in the same way. Each module's contents are organized in units, namely overview, genetic introduction, explanation, applications, typical mistakes, exercises, visualizations, information and supplements. By our didactical design those units are linked with an underlying model of competences as shown in table 1.

competence name	included skills and abilities	related module units
technical competence	calculate, draw graphs, …	information, exercises
comprehension	recognize and describe connections between concepts, …	genetic introduction, explanation
application and modelling	solve problems in contexts, …	applications
diagnosis of mistakes	find mistakes in mathematical argumentations, …	typical mistakes

Table 1: Competence model and related module units (see Fischer 2014, pp. 56-57).

In Göller, R., Biehler, R., Hochmuth, R., Rück, H.-G. (Eds.). *Didactics of Mathematics in Higher Education as a Scientific Discipline – Conference Proceedings.* Kassel, Germany: Universitätsbibliothek Kassel.
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The manifold units of a module serve to enable a variety of different learning approaches. The described, consistent structure of every unit makes it possible to postulate different learning approaches. VEMINT's learning scenarios make use of the different units and advise the use of a specific set of a module's units within a learning approach. An apparent approach would be to go through a module's core units step by step to learn the domain knowledge anew or repeat it thoroughly. Selective approaches include fewer units, like a training scenario (focus on exercises) or the use as a reference book (info unit) with focus on the relevant definitions and theorems only. Once an approach is identified, it can be used again in any other module due to their coherent structure. For a more detailed description of VEMINT's learning materials see Biehler et. al. (2012).

Possible answers for the challenge of self-regulated learning

Many years of experience with predominantly e-learning based blended learning courses confirm that the expectations at the learner's self-regulation competences are high. As a consequence students need to be supported in this regard. They need help with the selection of contents to focus on during the course. Hence, they are offered study plans for the course duration, which suggest a selection of modules (mathematical domains) they should work through with regard to their personal course of study, and also weekly work plans to orientate on for short term planning. Among the support features of the learning materials are the module structure and its units as well as the built in support to monitor one's personal learning progress. They can be found within each module and should be properly explained to the learner. So far only text based manuals were provided. The potential for improvement here was to get those explanations directly into the context where they are needed without creating a distraction from the contents and without splitting the learner's attention between the manual and the things that are explained. For this reason the interactive manual eVEMINT was developed.

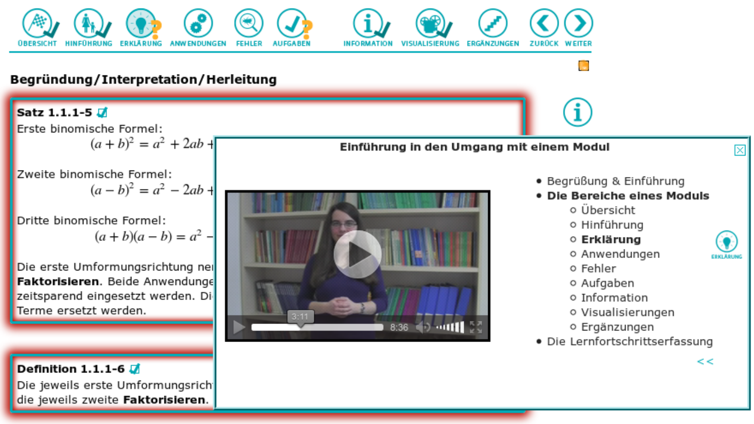

Figure 1: A screenshot of eVEMINT while definition and theorem boxes from a VEMINT module are beeing explained and highlighted next to it.

The interactive manual eVEMINT

eVEMINT is integrated into a module of VEMINT in such a way that it does not look like something 'extra' from the learner's perspective. In the overview unit of a module a button opens the interactive manual. Once it has been started, a small window within the module pops up. This is the actual manual. At the first glance it is a video with a person in it who explains the module and a table of contents to quickly navigate the manual next to the video. The interactivity starts to play a role while the video is playing. On the one hand it interacts with the user who keeps full control and can start, stop, move or close the manual at any time. On the other hand eVEMINT interacts with the module itself. Instead of iconic representations it brings up the parts of the module which are currently explained on the screen themselves. Important pieces are highlighted with a red frame. This way the learner can stop manually at any given time and experiment with the explained features and continue at will. You can take a look yourself at http://go.upb.de/eVEMINT (unfortunately only a German version is available). Already existing manuals have remained untouched from the introduction of eVEMINT and remain available for learners with deviating preferences.

The beneficial composition of design-paradigms

Essential for the conception of the interactive manual were the Cognitive Load Theory (CLT) from Sweller et al. 2011 and the Cognitive Theory of Multimedia Learning (CTML) from Mayer 2001. The principles derived from those theories are based on certain assumptions, but the effects were also empirical proven. For both cognitive theories the assumption of a working memory is vital. In the working memory incoming information are actively processed and also connected with information from the long-term memory. The space of the working memory is limited. Especially important for the CTML is the assumption of dual input channels to the working memory – the visual and auditive channel. We will exemplarily describe some principles and their implications on eVEMINT's design.

According to the principle of coherence from the CTML, it interferes with the learning process if irrelevant extra information is provided to the learner. This applies to (for the solution) unnecessary context information about a problem with an application context as well as to additional sounds like background music. According to the multimedia principle from the CTML, it helps to provide information on both channels at the same time. This is the reason why the interactive manual uses a video with sound to explain the module to the learners. S/he can focus visually on the module while listening to the explanation. Other sounds than the voice do not occur. The video only shows the speaker with a plain background (book shelf) to avoid distractions. The speaker herself is intended to create a positive emotional response with the learner and is thus not considered as unnecessary information.

A detailed explanation might be just right for beginners, but advanced learners do not need the same level of details and, on the contrary, they might be slowed down or disrupted. When this happens we talk about the reverse-expertise-effect from the CLT. The right amount of information depends on the addressed individual. This effect might occur with eVEMINT when a learner was already introduced to the learning material, e.g. in one of VEMINT's opening events. How to cope with this effect can be derived from usability engineering theories, which we will look at in the following paragraph.

From point of view with regard to computer science, multimedia learning materials are some sort of software with a graphical user interface. It cannot be argued that usability together with the look and feel of the materials are important and should not be neglected. Nielsen 1993 formulated ten usability heuristics for good user interface design. Although the heuristics seem naturally, they are often enough violated. One of the heuristics is that a program should contain shortcuts for experts – e.g. the key combination ctrl+s to quickly save a document. To include shortcuts for experts we added a table of contents to eVEMINT which is connected with the video and the module and thus allows precise navigation towards the specific parts for intermediate learners. Another usability heuristic recommends consistency of the software to the rest of the system. A simple example is the X-Button to close the interactive manual. This is a well known usability concept and its usage is intuitively clear. But also consistency to the learning materials themselves is important. eVEMINT uses the same colors, font type and graphics as are used within the module and due to its interactive approach and abstinence of iconic representation the representations of the learning materials are always up to date – even after content updates.

Selected poll results

eVEMINT has been piloted successfully during winter term 2014/15 bridging course. Figure 2 shows selected results from the latest evaluation in winter term 2015/16.

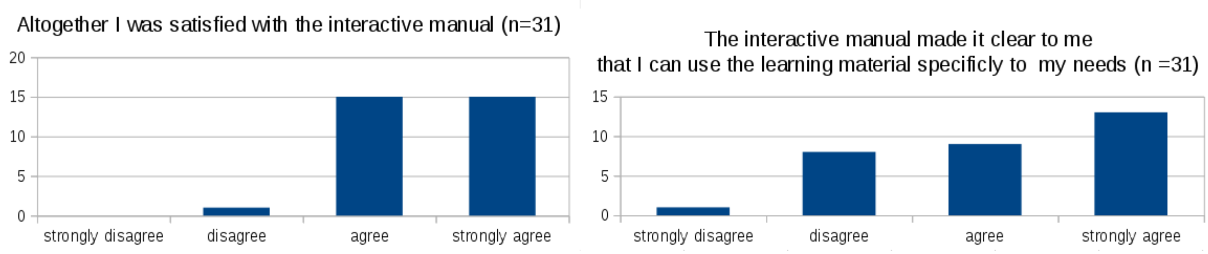

Figure 2: Student's opinions about the interactive manual.

It is safe to say that the interactive manual was very well accepted among the students who took part in the poll. It is also indicated that the participants have diverse needs. Some did not gain new insights from the interactive manual regarding personal directed and purposeful approaches, while others seem to have better understood and learned how the learning materials are designed to satisfy their individual needs.

Conclusion

A rich support of self regulation can be achieved to aid learners in situations of e-learning. However in sum they come for the price of increased complexity. We have shown that contextualized interactive manuals can do their part to make more complex didactic ideas accessible to the learners that certain design principles should be considered. A surplus of support offered without adequate guidance will not only likely be without effect, but can also increase the cognitive load and have a negative effect on the students' learning success.

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Didactics of mathematics in higher education, a service to science or a science in itself? Experiences made with tree--structured online exercises.

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Thousands of students need to be taught and tested in mathematics each year even though it is not their primary field of study. At RWTH Aachen, this situation led to the Didactics of Mathematics department developing tree-structured, adaptive e-learning exercises for a first-year engineering maths course. The ongoing project has been a challenge and a chance: On one hand, it has created an opportunity to conduct didactically oriented content analyses of higher mathematical topics, and high student numbers in the course have allowed for substantial empirical investigations. However, a separation of students into test groups has not been possible. In this text, we describe the exercise design process as well as some results of our evaluations up to now.

Introduction

Starting in the 2012/2013 winter semester, the Didactics of Mathematics department at RWTH Aachen University was asked to develop adaptive e-learning exercises for a first-year mathematics course of an engineering study programme. The motivation was to improve online learning opportunities to this course with roughly 1000 students, and in particular to help those among the students who had more serious problems like school-level maths deficiencies and/or unautonomous strategies relying on rote learning of sample solutions (Mustoe, 2001, p. 4; Rooch, Kiss, & Härterich, 2014, p. 399). For those learners, such deficiencies could then result in insufficient formal rigour, logical argumentation, and experience in solving tasks. The new e-learning exercises were therefore intended to feature

- direct and formative feedback (Shute, 2008, pp. 1, 177-181),
- high approachability / low threshold,
- structural overview,
- proximity to the process of solving tasks on paper (with regard to formal, logical, and heuristical aspects).

We opted to design them in a way such that users could "explore" exam-style task solutions. Such a design needs to offer multiple ways of approaching a problem, and it needs individual, adaptive feedback that responds to user choices.

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In order to realise this, we decided to include answer-dependent branching in the exercises, leading to a tree structure. Although the core idea – intra-exercise branching into different paths depending on the learner’s answers – goes back to Norman Crowder’s branching system type of programmed instruction (Lockee, Moore, & Burton, 2004, p. 550), our approach appears yet to be rare in mathematics e-learning. Contemporary mathematics e-learning platforms (like the European platforms “Math-Bridge” and “MUMIE”, or the Singaporean “ACE-Learning”) seem to either offer collections of unbranched questions or roughly follow the more linear approach originally proposed by B. F. Skinner, where a certain error rate in one question set sends the learner to a remedial question set in order for him to acquire more practice (Lockee et al., 2004, p. 547). At any rate, the tree structure has allowed for individual feedback and, in many cases, for exploring the multiple ways which are often possible for solving a task. Since winter 2012, the exercises have been used in the course every semester, gradually been extended, and been evaluated statistically as well as with student surveys. In the following, we will describe our approach in developing the exercises, and then present some of the results from the evaluations carried out up to now.

Design and development of the tree-structured exercises

What was the motivation to introduce new e-learning exercises in the first place? The mathematics course in question already had a diverse range of optional and non-optional practice elements, but it was found

- that the mandatory online test exercises (necessary for exam application) were too simple compared to actual exam tasks, while the optional harder written assignments (corrected by tutors) left weaker students clueless, which in turn made them resort to rote-learning of sample solutions;
- that especially the online tests and tutorial lessons could not give much individual feedback to students about what they were doing wrong and how to improve;
- that weaker students could often hardly be convinced to learn autonomously and to try to solve tasks without sample solutions.

Our new e-learning exercises should address these issues. The mathematical contents of the course in question had previously been established in collaboration with the engineering departments, so these contents were a fixed guideline to us. Since then, our work has been

- to select those types of exercises that are deemed problematic for some learners, often found out by direct experience with students;
- to perform a didactically oriented content analysis on these topics, finding aspects important for understanding them – supported by experience with students (the developers were also working as tutors in the course) as well as observing one's own thought processes while solving an exercise;
- to transform these findings into adaptive e-learning exercises which are close in style to exam tasks, which give individual feedback to at least some extent, and which offer enough coaching elements to activate even less well-performing students.

As mentioned, we have decided to design our exercises in a tree-structured way in order to reach the goals above. How could such answer-dependent branching be realised technically? The choice fell on the "lesson" feature of Moodle: It offers consecutive linking from one page to one or more others, either knowingly chosen by the user on a "content page" (an example is shown in figure 1) or determined automatically by the answer given on a "question page".

This makes it possible to follow different paths leading to a solution, and to give different feedback according to what the user has answered. We call this intra-exercise branching "local adaptivity", as an exercise can react directly to user choices/answers from one page before, but not indirectly to any actions before that (it does not store behaviour in variables).

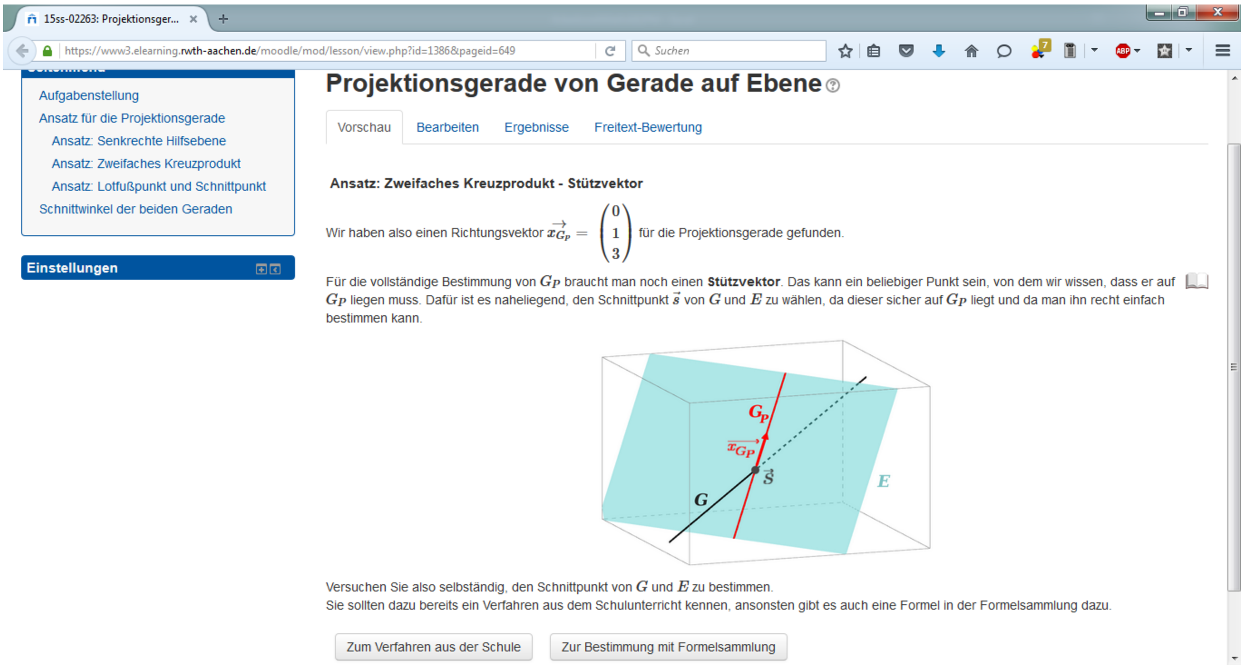

Figure 1: A content page in an exercise on analytic geometry, where the user can click link buttons to choose the method for determining an intersection point. Note the overview menu in the upper left and the "look up this concept" book icon at the far right. Similar icons exist for "write down" and "pay attention".

Content-wise, the exercises have been designed to follow the solution process of exam-like tasks such as a principal component analysis, solving a separable differential equation, etc. Every tree-structured exercise starts with such an exam-like task description and then follows the corresponding solution process (or several alternative solution processes) step by step up to a solution. It is important that this solution process also includes intermittent thoughts and/or an initial heuristic trial-and-error phase, not only the final phase of writing down an actual neat solution.

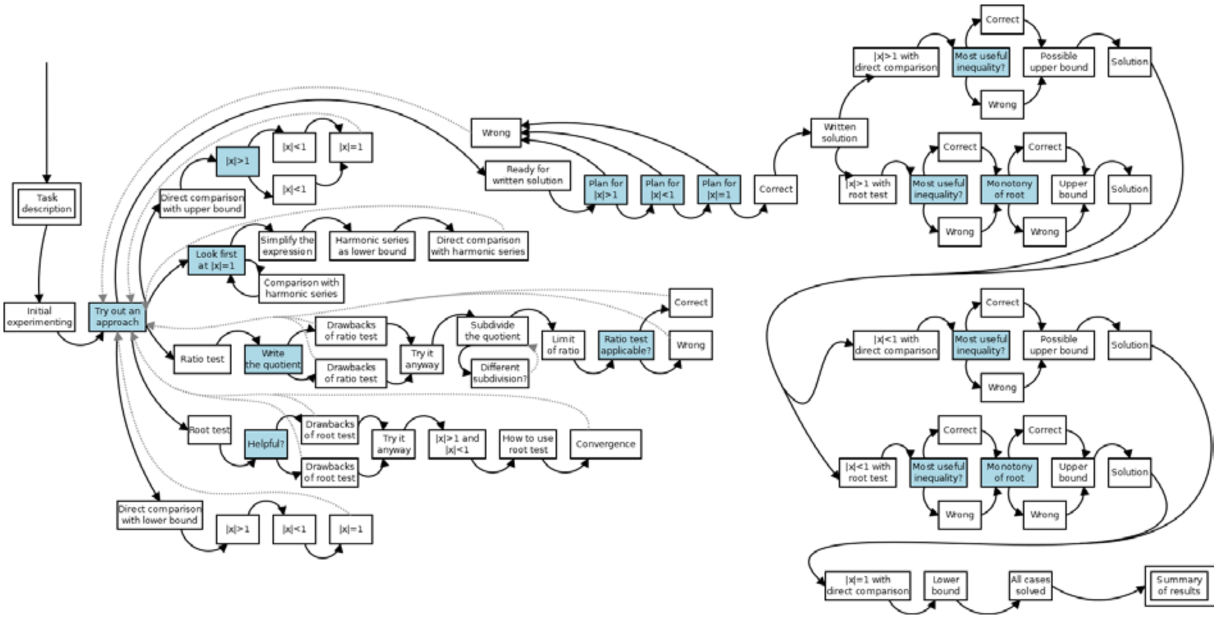

Figure 2: Structure of a series convergence exercise.

In figures 2 and 3, one can see the structures of two example exercises; the first one requires the user to examine the series

$$\sum_{k=1}^{\infty} \frac{\sin(x^{6k})}{k + x^{2k}}$$

for convergence depending on x from the real numbers, the second one asks for all real solutions to the differential equation system

$$\vec{y}\,' = \begin{pmatrix} 4 & 5 \\ -1 & -2 \end{pmatrix} \cdot \vec{y}\,.$$

Figure 2 shows how a large portion of the series convergence exercise – the whole left side in the diagram – is reserved for an initial trial-and-error phase. With the help of explanations on respective pages, the user tries out different convergence tests for applicability. When the user feels ready to start writing down the actual solution, he has to correctly answer which convergence / divergence test can be used for which x (in the three blue consecutive question pages in the upper top portion). The right side of the diagram then illustrates the solution writing phase. Here, the user is led through the solution for three different cases for x (with a couple of question pages); and as shown in the diagram, two cases can each be solved in two different ways.

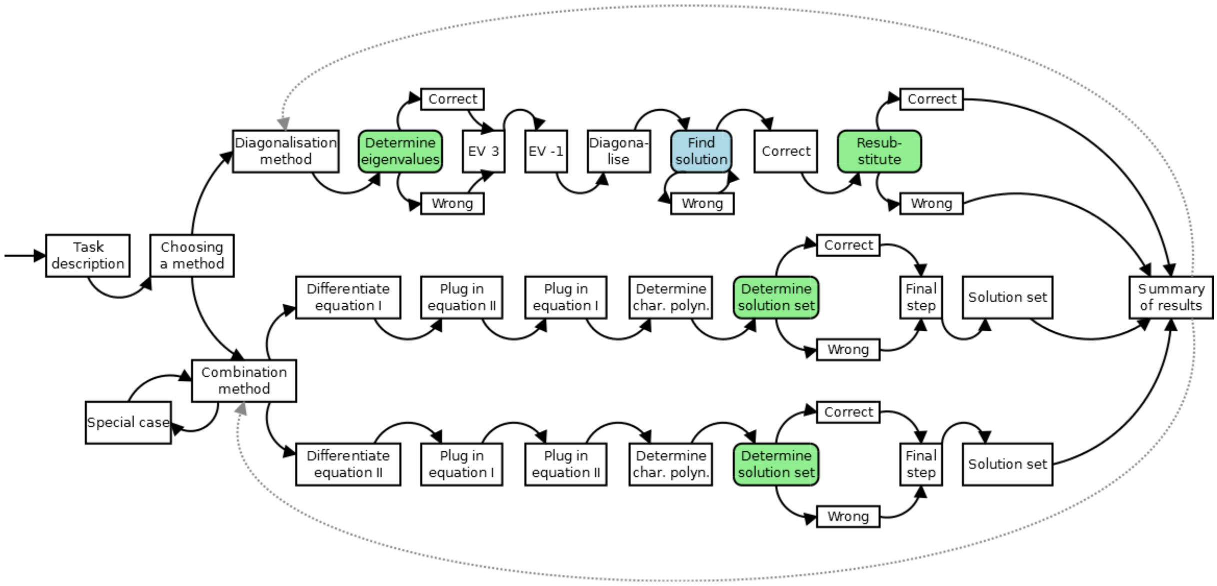

Figure 3: Structure of a simple differential equation system exercise.

The structure of a shorter exercise on a 2×2 linear differential equation system is shown in figure 3. Again, there are different ways to reach the solution, with the top one using the matrix diagonalisation method, and the two lower ones using a "combination" method (differentiating one of the two equations and smartly combining all the resulting equations), which can here be performed in two different but similar ways.

What are typical aspects found in a didactically oriented content analysis to such exercises?

- In the series convergence exercise from above, some aspects important to understanding can be: Knowing common convergence tests; recognising when a specific convergence test is useful (as simple examples, the quotient test might solve some types of fraction expressions, the root test might solve some expressions to a power of k); recognising asymptotic behaviour of an expression; developing an instinct on which cases of x need to be distinguished (for series with a variable x); knowing common inequalities like the triangle inequality; thinking from the back to the start and knowing some common tricks (for constructing inequality chains).
- In the differential equation system exercise, aspects (apart from obvious necessities like factorising quadratic polynomials, diagonalising a matrix etc.) can be: Understanding the difference between the two methods available, especially knowing in which cases they can be used; understanding that for a specific solution, the parameters from the solution set have to be chosen in correct correspondence in the upper and lower component; being aware that different methods can lead to slightly different-looking but equivalent expressions for the set of solutions.

For each topic, we have tried to include most or all of those aspects in the corresponding exercise(s), for example on explanation pages or, when possible, on correction pages following a common mistake.

In conclusion, our exercises can be seen as interactive, explorable solving processes. They always include detailed thoughts on every step, question pages for intermediate results, one or more sample solutions, and sometimes trial-and-error phases. We want to stress that, more than just a rote learning of solution patterns, learning with sample solutions can indeed be effective when enhanced as described above (also cf. Ableitinger & Herrmann, 2013, p. 27).

Evaluation results up to now

The engineering mathematics course in question is a two-semester course with one exam each semester. Following some content changes in the exercises, as of mid-2016, there are 11 tree-structured exercises for the winter semester and 8 exercises for the summer semester. The percentage of students who have used our tree-structured exercises varied over the semesters – roughly half of all students in some semesters, or just around a quarter of all participants in other semesters. This difference is likely due to changing ways in which the exercises have been included in the course.

One part of our evaluations has been a student survey. Starting in autumn 2013, we have conducted a survey after the end of each full cycle, with largely positive results every year (although suryey participation numbers have been low in 2015): Roughly 80% of survey participants each year have found the tree-structured exercises helpful for understanding the mathematical concepts and using them in tasks. Around 40% have even stated that without the tree-structured exercises, they would probably have performed worse in their exams. A common remark has been that – while our exercises are helpful – written assignments are still a better preparation as they are closest to exam conditions.

The second part of our evaluations up to now has been the analysis of correlations between exercise usage (optional tree-structured exercises, optional written homework, mandatory short online tests) and exam performance. This analysis has been conducted every semester on the basis of anonymous student data; the investigations for every semester from winter 2012/2013 until winter 2015/2016 have been completed. Among others, we used Pearson correlation coefficients for the correlation between the number of points reached in the exam and

- the number of optional tree-structured exercises used;
- the number of points reached in the optional written homework assignments;
- the number of points reached in the mandatory short online tests;

with all of these correlations being positive. Participants who later decided not to take the exam were counted as zero points. While the concrete values for these correlations have varied over the semesters, their relative standing to each other has remained rather stable. This means that the written assignments have in most cases shown the highest correlation (from around 0.3 up to around 0.5), with the tree-structured exercises' correlation being slightly lower (also from around 0.3 up to around 0.5), and the mandatory online tests' correlation being the lowest (from around 0.1 up to around 0.3). These results also reflect the students' estimation of the helpfulness of our exercises.

Conclusion

Coming back to the title question, we hold that didactics of mathematics in higher education can be seen both as a service *and* a science: From a scientific point of view, developing learning aids for university lectures is an opportunity to open the field of higher mathematical contents to didactically oriented content analysis, which has mostly been reserved for school mathematics before. It can also serve as a “testing ground” for new approaches, as in our project, and high student numbers allow for statistically sound empirical investigations of effectiveness. It should be noted, however, that fairness constraints often prohibit test group separation for statistical hypothesis testing – as is the case with our course in question. From a service point of view, cooperative projects often result in didactically well-designed products, usable for a long while, and which can be helpful for (weaker) students.

Finally, specifically concerning our tree-structured exercises: Even though written assignments can still be suspected to be the most effective option for learning and exam preparation, the fact that the tree-structured exercises’ correlation is rather close to the written assignments’ correlation is certainly interesting and speaks for a certain effectiveness of the exercises.

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The use of digital technology in university mathematics education

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Since the advent of personal computers in the 1980s various teaching scenarios have been created in which computers were used to enhance understanding. Since modern computers are capable of performing computations very fast, one nowadays can create explanation material in which the effects under considerations emerge from the combination of basic (rather elementary) building blocks. The article discusses scenarios under which these types of simulations and visualizations can be profitably used in teaching and explaining. The article also discusses some quality criteria that must be applied to the tools used to create such visualizations.

Why computers?

It is almost a commonality that computers are a versatile tool for enhancing comprehension in teaching situations. Nevertheless it is appropriate to ask from time to time „why“ once should use computers in teaching scenarios of different types. In particular one might ask whether there are particular situations in which *only* a computer could transport a certain learning experience to the student. From a birds eye perspective the abilities that are specific to modern computers can be divided in the following categories:

- perform computations very fast,
- provide a viewing surface that can be updated very fast and on demand,
- provide several types of input channels for interaction,
- give access to huge amounts of stored data including different kinds of media,
- create various types of communication channels to other users.

We here will focus on teaching scenarios related to the combination of the first three items neglecting those scenarios that take advantage of the storage and database aspects (access to films, pictures, lexicons, etc.) and the social aspects (communication, interaction within groups, forums, etc.).

The first point (fast computations) should not be mistaken as as mere possibility of doing fast routine number crunching: Fast computations give the opportunity to create *complexity* (in combination with appropriate software concepts). In particular, it becomes possible to create simulations based on first principles and observe certain high level structures as emerging effects (consider for instance a physics simulation that illustrates the conformation of molecules based on the mutual interaction of single atoms). The second point (viewing platform) provides the opportunity to visualize the results of the computations. If computations and visualizations are fast enough this creates a cinematic experience of a process or

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effect (to stick with our example, one can see how the atoms under their mutual forces start to form molecular structures). Finally the third point (interactivity through input channels) provides the possibility to interact with the simulation. Provided everything is fast enough a situation is created the exceeds the purely cinematographic experience. One may get an immersive interaction within a simulated *micro world* (in our example: playing „god" by moving single atoms around and see what happens) (compare Richter-Gebert 2013).

What are the benefits of these capabilities for teaching? Using the simulation power of a computer gives the student (both on school and university level) the possibility to experience a certain context in a very direct (and ideally intuitive) manner. By this the student can become a „researcher" that explores a context and gets involved in a very direct manner. Such considerations not only form the basis of dynamic geometry software (like Cabri, Geometers Sketchpad, GeoGebra or Cinderella) but also many other more general mathematical visualization attempts as they may be found for instance in the collections of the Wolfram Demonstration project (Wolfram Research Inc., 2007-) , the collection of Mathe Vital (Richter-Gebert et al., 2007-) or on GeogebraTube (Hohenwarter et al., 2011-).

Usage scenarios

Let us use the adjective *highly interactive* for those interactive visualizations and simulations that are based on first principles and derive their effects as emergent structures from elementary building blocks. They are in contrast to mere *stimulus response animations* in which the computer only reacts according to narrow predefined patterns created by the programmer. We will briefly outline some scenarios in which the use of highly interactive visualizations provides a particular benefit.

Versatile demonstration material

Ideally a good simulation models a part of the abstract mathematical realm and maps it to a tangible entity on a computer screen. By this such simulations play a role similar to classical 19th century mathematical geometry models whose purpose was to make abstract mathematical concepts „tangible" and „visible". Digital simulations (if done well) may by far exceed the possibilities of physical models. They allow for a broad variety of interaction with the object under consideration. By this they become very versatile tools for teaching allowing a lecturer do demonstrate various aspects of the subject that should be explained. In the collection Mathe-Vital we aimed to create such digital-models for teaching scenarios for undergraduate university studies. All together over 500 such visualizations were written so far covering different branches of mathematics and a huge variety of objects. To demonstrate the spectrum of these models we here just mention a few topics that were created for a Linear Algebra class: Finite additive and multiplicative groups, modulo arithmetic, Euclids algorithm, continued fractions, transformation groups, stereographic projections, vector arithmetics, eigenvectors, linear differential equations, and many more. Each of these demonstrations was created to support a certain teaching situation with interactive visual material that allows for a flexible usage within a specific context.

Self studies

The models mentioned above can also be used in a scenario in which a student wants to go deeper and explore the effects by his own. To stay with our comparison to physical models this corresponds to the situation in which the lecturer has explained a model and then passes it around in the class (or even allows your students to take it home). Experimenting with a specific *digital micro laboratory* (compare Richter-Gebert 2013) can set the student in the situation of a researcher who explores how a certain predefined scenario behaves in specific circumstances. For instance in the examples of atoms that form molecules under their mutual forces one might be interested to see what happens if the atoms have different charges, masses, etc. In the example of modulo arithmetic and subgroups of finite multiplicative groups one might explore which sub-groups are generated by various collections of elements.

Learning by creating worlds

Taking this one step further one might even create situations in which students themselves create simulations or micro laboratories. Putting the students in the role of a model-maker gives them a chance to explore a topic in depth and at the same time thinking of the educational value of the model they create. Perhaps this is best expressed by the words Felix Klein put it in his *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhunderts (1928).* He describes the role of models in the 19th century geometry school as follows:

> *Wie heute, so war auch damals der Zweck des Modells, nicht die Schwäche der Anschauung auszugleichen, sondern eine lebendige, deutliche Anschauung zu entwickeln ein Ziel, das vor allem durch das Selbstanfertigen von Modellen am Besten erreicht wurde.*

Almost literally this description applies to the creation and use of digital models if they are created by a highly interactive first principle approach.

Tools for creation of digital mathematical content

Creating digital models for a specific mathematical context is by no means an easy task and generally requires programming skills as well as a deep mathematical understanding as well as a good „gut feeling" for the educational value and implication of a specific model. During the last three decades several tools for the creation of such models have been released. They have different origins (like computer algebra, dynamic geometry, physics simulation engines, raytracing,...) however nowadays many of them provide methods to create mathematical content on a fairly high abstraction level. In particular a certain convergence of concepts can be observed. While computer algebra systems (like Mathematica or Maple) are more and more enhanced by interaction and visualization capabilities on the one hand, dynamic geometry software, on the other hand, is more and more enhanced by scripting languages, and computer algebra components. For such tools to be useful it is important that they satisfy several quality standards since they will have an direct impact on the educational value of digital models created with them. We only list a few of them

- Many mathematical objects (like points, lines, vectors, polynomials, types of numbers,...) and operations on them should be accessible, on a high level of abstraction.

- The behavior of such objects and operations (even of very elementary ones) should be correct and free of artifacts on a high mathematical level. This usually requires a significant amount of mathematical modeling and a deep understanding of the elementary objects and operations (see for instance Kortenkamp & Richter-Gebert 2001).
- Despite the mathematical depth the objects should be accessible in an easy and understandable way to the user of the tool.
- There should be a close semantic correspondence between the abstract objects, their visual representation and their interaction pattern.

Similar quality criteria apply on another level when concrete interactive visualizations are designed with such tools. A detailed analysis of this topic together with crucial aspects relevant for the design are given in (Richter-Gebert 2013) and (Richter-Gebert 2015).

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Rethinking refresher courses in mathematics

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The stated purpose of refresher courses in mathematics at many German institutions for higher learning is to offer incoming students the opportunity to bridge the gap between their actual mathematical skills and those expected of them. Several studies have supported the hypothesis that the students with weak skills tend to complete at most a small portion of such courses. This contribution provides a brief overview of how some programs attempt to meet this challenge. A pilot project at the University of Applied Sciences Ostwestfalen-Lippe that combines tutoring and mentoring within and beyond the refresher course is presented along with initial findings.

Extended Abstract

Many colleges and universities in Germany offer refresher courses to incoming engineering students as a review of school mathematics. The main goals for these courses on the meta-level are often to reduce attrition rates, to reduce the heterogeneity of student knowledge, and/or remediation (see for example Bausch et al., 2014). Unfortunately, the time allotted is often too short for these purposes. Furthermore, it is not clear if the intended audience is being reached (see for example Roegner, 2012). Moreover, the effectiveness of such programs with respect to these goals is often difficult to measure. The question of how to reshape refresher courses with measurable goals in mind arises.

The refresher courses at the University of Applied Sciences Ostwestfalen-Lippe (HS OWL) are similar in nature to other such courses in Germany. They run for two weeks prior to the beginning of the semester. Most participants have just recently completed their schooling. The instructors are often retired schoolteachers or master's students. The topics to be covered are vast so that a lecture style prevails. Attendance at the beginning of the course ranges anywhere between 30% and 70% of the students enrolled for their first semester. Attendance at the end amounts to a handful of students who seemingly could have refreshed their skills during the semester. So why even bother with refresher courses?

Inspired by the socialization and independent learning aspects in certain US and English programs (e.g. Tutoring Lab at Eckerd College, Math Lab at Southeastern Louisiana University, Mathematics Learning Support Centre at the University of Loughborough) and the utilization of "learning-scouts" at the HS OWL, an alternative approach towards the refresher course was conceived and is currently under development. The main idea is to reduce the content, allowing more time for the participants to solve problems. The students are supported in this phase by a learning-scout: a student in the same field of study as the participants and trained in the principle of minimal help. The focus shifts in this way from quantity to quality. The participants not only build a social net with other new students, they also become acquainted with someone who has successfully completed the first year(s) of study. Thus the

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learning-scout also acts as mentor and role model. This learning-scout supports the first-semester mathematics course so that the students participating in the refresher course (hopefully) not only stay longer in that course but are also more motivated to use the resources offered in the first-semester math course during the self-study phase.

In the pilot phase, a learning-scout was placed in one of the refresher courses at HS OWL in Höxter prior to the Winter Semester 2015/2016. The arrangement made with the lecturer was basically "business as usual". It was, however, agreed upon that some time would be allotted each day for problem solving. During this time, the learning-scout would assist the students. Observations concerning the interactions between participants, lecturer, and learning-scout were recorded in order to provide a foundation for future studies. As expected, the students interacted with each other for longer periods of time when the learning-scout supported any one of the given students. According to the students in the parallel session (without a learning-scout), there was neither time for the students to actively solve problems nor time to interact with each other.

First findings of this preliminary study are given that are relevant to the alternative set-up that is to be piloted in the Winter Semester 2016/2017. After a detailed description of the proposed set-up, comments, criticisms, and suggestions from the audience will be welcomed.

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Innovative education in mathematics for engineers. Some ideas, possibilities and challenges

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At the Norwegian University of Science and Technology (NTNU) extensive work is being done to change the way basic mathematics is taught to engineering students. The focus is being shifted to more student centred approaches to teaching and learning and the aim is that students should develop deep understanding of mathematical concepts and processes. Various steps have been taken to stimulate students' own involvement and activity in the learning process. Such steps include changing the format of the lectures, providing more 1-1 contact with students and a diverse selection of learning resources. Resources are provided on digital platforms both in the form of videos and as computer aided assessment. This paper reports on experiences from the first two years of the project.

Background

The Norwegian University of Science and Technology (NTNU) is the main institution for education of engineers on master level in Norway. Every year some 1600 students are admitted to 18 different study programmes in engineering. Due to high minimal requirements for admission as well as high competition to many of the study programmes, most of the students have a very strong background in mathematics when they start studying. Despite this fact one has faced rather high failure rates in the basic mathematics courses as well as high drop out rates from some of the study programmes.

It is recognised that although the students have had to take as much mathematics in upper secondary school as possible with rather good grades to be admitted, and therefore could be expected to have a strong interest in mathematics, their main motivation for seeking engineering education is not to study mathematics as such. Therefore they are expected to come with varying motivation for learning mathematics and it is a challenge to organise the education in ways that can meet all students' needs.

The high number of students and relatively limited teaching resources has traditionally been handled by giving lectures in large groups complemented by problem sessions in small groups led by student assistants, sometimes just one year ahead of the students they are tutoring. The project *Quality, accessibility and differentiation in the basic teaching of mathematics,* launched in 2014, with a pilot starting in 2013, has as two of its main goals to increase differentiation and to enable closer contact between students and highly qualified teachers. Further goals are to stimulate students to increased and more continuous work input in the studies and to increase student activity and involvement in the education.

The project covers the courses Calculus 1 (autumn) and Calculus 2 (spring). Calculus 1 is given to all the engineering programmes and Calculus 2 to almost all. Some programmes in computer science take a course in discrete mathematics instead of Calculus 2.

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Some important components in the project

To achieve accessibility and differentiation a guiding principle has been that various types of learning resources should be made available when and where students need them. Therefore learning material is available on the internet in the form of web pages and videos of different types. Videos range from simply capturing a regular lecture or a problem solving session and put it on the web to producing more focused thematic videos dealing with one particular concept that is central, and perhaps also known to be difficult to get the grip on. A student support centre (drop-in centre) is open every day from 12.00 to 18.00 (Fridays to 16.00). Here students can come as they wish to work on their own or in collaboration with fellow students and there are always teachers there that can be asked for help. The main responsible persons at the drop-in centre are usually PhD students in mathematics. They are assisted by students with lower qualifications and also the regular lecturers are present at the drop-in centre at given times of the week. The drop-in centre is a component that addresses both quality, accessibility and differentiation, quality in the sense that one aims at using teacher with high mathematical qualifications there.

In order to shift teaching resources towards using more highly qualified personnel some of evaluation tasks have been moved over to the computer. Traditionally students have handed in paper-based solutions to problem sets every week and these solutions have been assessed by a large number of, often low qualified, student assistants. The weekly problem sets are still maintained but now they are done on the computer, using a computer aided assessment system (Maple T.A.). Paper based hand-ins are limited to one per month, comprising more elaborate problems and where the students can expect to get more helpful feedback which will bring them further in their learning process.

Theoretical background

The traditional teaching of mathematics at university can be seen as based on a metaphor for learning that entails transmission of knowledge from those who know to those who do not know. It is widely accepted that in order for learning to take place some kind of activity on the learner's side has to take place. Also it is well recognised that participation and communication are important factors to foster learning (Lave & Wenger, 1991). Although lectures can certainly involve student activity they may also lead to passitivity and in order for lectures to function well it is necessary to shift the focus from teaching to learning and instead of producing a best possible presentation to focus on how the presentation can enable the students to learn in a best possible way (Chang, 2012; Engelbrecht & Harding, 2005). The project is based on a view on knowledge and learning that recognises students' engagement as essential for a good learning outcome. This includes both an affective, behavioural and cognitive engagement (Fielding-Wells & Makar, 2008).

In recent years the idea of 'flipped classroom' (Mazur, 2012) has attracted a lot of interest, also at higher education. In order for this to function well it requires a certain engagement on the students' behalf in terms of preparing for classes. Previous research indicates that this might not be successful with students who are not used to it (Sopasakis, 2013). Challenges might be expected to be even higher when, as is the case at NTNU, one is facing rather large student groups in one classroom. However, there is evidence to show that even in large classes this approach may be successful (Deslauriers, Schelew, & Wieman, 2011).

Findings from the project

In order to monitor the project surveys have been carried out each semester since autumn 2013 (two cohorts of students) with approximately 700 respondents in Calculus 1 and approximately 600 respondents in Calculus 2. In addition focus group interviews have been carried out with a small number of students every semester and the students also have the possibility to give anonymous feed back through a web-based application every week (web diary).

Some of the questions that have been asked in the project concern how students work with mathematics, what learning resources they use and how they value the available resources. By using many of the same questions in the survey each time one will be able to see whether there are any changes over time.

When presenting the students with a list of available resources one can detect a certain preference for the traditional learning resources. About 70% report that they “to a large extent” attend lectures. About 20% watch the video recorded lectures “to a large extent” and other kinds of videos are used by a much smaller portion of the students. The lectures are also highly valued as a source for learning, about 40% agree “to a large extent” with the statement “I learn a lot by going to lectures”. Another 40% agree “to a rather large extent” with the same statement. In interviews students have been prompted to elaborate on the role of lectures and this reveals that lectures play different roles in the students’ life. On important role is that they provide some structure in the students’ daily life (“Lectures make me get up in the morning”) and they also provide a sense of good conscience (“When I go to lectures I feel that I have done something useful”). Reasons that are more directly linked to the learning process are that the lectures provide a natural arena for discussing with fellow students and also for talking to the lecturer. Lectures are given in two times 45 minutes with a 15 minutes break and this break is used eagerly to ask questions to the lecturer. There is also evidence to suggest that mathematics may be somewhat different from other subjects in the sense that it is hard to read mathematics from a book. In interviews students come up with statements like, “in mathematics it is important to have things *explained* to you”.

The textbook is also placed in high regard by the students, 80-90% report that they use it “to a large extent” or “to a rather large extent”. When being asked about how they use the textbook it seems that the most important role of the textbook is as a source of help when the students are stuck on a problem. They will then use the textbook to look for an example that resembles the problem they are working on. The theory in the textbook seems to play a less important role but there are indeed about 75% of the students that agree, completely or to some extent, to the statement “I read the theory in the textbook”.

The traditional lecture structure allows the students to be rather passive and changing to more student active structures will require a higher degree of involvement on the students’ side. This could be challenging since there is evidence to suggest that students do not prepare for lectures to a large extent. Our surveys indicate that around 80% never or rarely read "today's topic" before lectures and around 55% never or rarely work on the material the same day after the lecture. These results, as well as results from other projects (e.g. Sopasakis, 2013), suggest that one should approach the changes in the traditional structure with some care.

Survey lectures and interactive lectures

In the traditional structure of Calculus 1 lectures have been provided as 2 times 45 minutes slots two days a week. Students are grouped in six sections, according to their study programme. This structure will usually lead to a style with basically one-way communication and the students spend the time listening and taking notes. From the start of this academic year a new structure has been implemented. The first 2 x 45 minutes session of the week is announced as a 'survey lecture'. This is given in a room of maximal size (400 persons), four times to accommodate all students, and it is also video taped. The purpose of this lecture is to give an overview of the material that the students are supposed to work with during the coming week. The second session of the week is announced as an 'interactive lecture' for which the students are given problems in advance to prepare for class. During the interactive lectures the students can work on the given problems with assistance from the lecturer, and also on other problems that the lecturer may present during the lecture. The interactive lectures are given in 12 sections, i.e. still a rather high number of students in each section. An electronic student response system is used to generate activity and communication during the lecture. In the last phase of the interactive lecture the teacher uses the input he/she has acquired during the lecture as the basis for discussing main issues of the week's topic. In a quick survey done after about five weeks into the semester 80% of the respondents (n = 462) report that they benefit from the interactive lectures to a large or rather large extent and around 75% report in the same way about the survey lectures.

Preliminary conclusions

The project is still running and it is too early to report on any effects or impact from the project. A general impression from the findings is that students have a preference for traditional methods and that therefore changes should be implemented with care.

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The role of mathematics in the design of engineering programs – a case study of two Scandinavian universities

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A recently initiated project is introduced, focusing on mathematics within engineering education from an institutional perspective. The aim of the study is to identify and investigate conditions and constraints for the teaching and learning of mathematics created through the organizational structure of engineering programs. The study is a case study of two engineering programs at one Norwegian and one Swedish university, and it is set within the overall framework of the anthropological theory of the didactic (ATD). So far, very little data has been collected, and only some initial observations can be made. From these observations, however, some indication of an applicationist understanding of mathematics within civil engineering education at both universities can be found.

Introduction

In recent years, university mathematics education (UME) research has taken an increased interest in the teaching of mathematics to non-mathematics majors as evidenced, for instance, by a number of papers presented in the UME working group at the 9th Congress of European Research in Mathematics Education (CERME 9) earlier this year (Nardi et al, in press). The teaching of mathematics in engineering programs is of particular relevance, given the large number of students enrolled in such programs, the perceived importance of mathematics in engineering, and engineering students' reported difficulties with mathematics courses they are expected to take, leading for instance to high attrition rates (e.g. Alpers et al 2013, Broadbridge & Henderson 2008).

This paper introduces a recently initiated project focusing on mathematics within engineering education from an institutional perspective. The aim of the study is to identify and investigate conditions and constraints for the teaching and learning of mathematics created through the organizational structure of engineering programs, as exemplified, for instance, by program and course syllabi, forms of assessment, organization of teaching etc. An example of such constraints presented in the literature is the notion of 'applicationism', as introduced by Barquero, Bosch and Gascon (2011). In their words, applicationism is an understanding of mathematics and its relation to the natural sciences where "first mathematical tools are built within the field of mathematics and then they are 'applied' to solve problematic questions from other disciplines, but this application does not cause any relevant change, neither in mathematics nor in the rest of the disciplines where the questions to study appeared." (ibid, p. 1940) Barquero, Bosch and Gascón see the epistemology of applicationism that they detect in various mathematics syllabi and textbooks in Spanish universities as an obstacle to the implementation of modelling activities in the mathematics courses taught in science programs. Similarly, the prevalence of applicationist tendencies in the syl-

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labi of engineering programs might help explain reported difficulties in implementing, for instance, integrated curricula (e.g. Merton, Froyd, Clark & Richardson 2009).

The main focus of the study will be on the mathematics courses within the programs, but there will also be consideration of the role of mathematics in the courses in other subjects. Anecdotal evidence from colleagues involved in engineering education and other service teaching of mathematics suggests that the way mathematics is talked about and used in the students' main subjects can have a significant effect on the way they approach the study of mathematics. There is ongoing research in various areas relating to this issue. For instance, there are studies on the mathematics needed for problem solving in general engineering contexts (Biehler, Kortemeyer & Shaper, in press; Hochmuth, Biehler & Schreiber, 2014; Lehmann, Roesken-Winter & Schueler, in press); on differences in how mathematical concepts are presented in textbooks in engineering and mathematics (Alpers, 2015); and on differences in instructors' attitudes towards the teaching of mathematics to engineering students, depending on their academic background (Hernandes Gomes & González-Martín, in press). Studies like these all suggest the relevance of considering not only the position of the subject of mathematics in engineering programs, but also the position of mathematical content and practices within other subjects in the engineering programs.

Theoretical framework

The project is set within the overall theoretical framework of the Anthropological Theory of the Didactic (ATD). Central to ATD is the notion of praxeology. According to Chevallard (2006, p. 23) a praxeology is "the basic unit into which one can analyse human action at large". In a praxeology, "two inseparable aspects can be distinguished: the block of the *practice* (or *praxis*) that consists of *types of problematic tasks* and *techniques* to tackle these tasks. Linked to this block, a reasoned discourse (*logos*) arises about the practice, whose function it is to provide a description, explanation and justification of the practice." (Barquero, Bosch & Gascón 2012, p.312, italics in original). The logos is in turn made up of *technology*, the discourse on the technique, and *theory*, providing the foundations of the technology. Praxeologies can be classified according to increasing complexity, starting with specific praxeologies, built up around a single type of problem, which are then "linked according to their theoretical background to give rise to local, regional or global praxeologies" (Bosch & Gascón 2006, p. 59). In engineering education, from the perspective of this study, two different global praxeologies can be distinguished – a mathematical and an engineering praxeology, with different tasks, techniques, technologies and theories. Still, mathematics plays an important role in engineering education, and an investigation of the role mathematical techniques and technologies play in the engineering praxeology might be a fruitful way of investigating questions relevant to the aim of this project.

Another aspect of ATD relevant to this project is the notion of a hierarchy of "levels of didactic co-determination":

Civilization↔Society↔School↔Pedagogy↔Discipline↔Domain↔Sector↔Theme↔Subject

(Barquero, Bosch & Gascón 2011, p. 1938). These different levels, ranging from the most general – civilization – to the most specific – the subject of study in a particular teaching situation – all affect one another, providing conditions and constraints for how teaching and learning activities can be organized in a specific context. Given that the present study aims to focus on how the organizational structure and the design of engineering programs affects the role of mathematics in the programs, the levels of didactic co-determination ought to provide useful tools for analysis.

Data collection and initial observations

The study focuses on the cases of two engineering programs, civil engineering and electronics, at one Norwegian and one Swedish university, and the main focus lies on the first two years of study (where most of the mandatory mathematics courses tend to be situated), although the overarching structure of the programs, for instance general learning outcomes of the programs, is also considered. The two universities chosen are similar in many respects – relatively new institutions of approximately the same size, with a mainly regional student recruitment base. At the same time, there are pronounced differences between them regarding, for instance, the role of examination, where the Norwegian university routinely uses external examiners for its courses, in contrast to the Swedish case where all examination is done locally. Also, engineering education in Norway is subject to a certain amount of governmental control through a national curriculum for engineering education issued by the department of education. Similar documents do not exist in the Swedish context, where there are only general national curricula for bachelor and master education.

Since at the time of writing the autumn semester is only a few weeks old, I have so far only been able to collect course syllabi, program descriptions and curricula, and the different national curricula. I intend to complement these with some sample textbooks and examples of homework and examination tasks, as well as information about what forms of teaching and learning activities are used in different courses. An analysis of textbooks should give insight into how the relation between mathematics and engineering is articulated there. This is likely to affect how students experience this, since in both the Norwegian and the Swedish context the textbook is one of the most important resources that lecturers use when planning their teaching. Homework and assessment tasks should give insight into the types of tasks and techniques in the different praxeologies, and also to what extent what is being said in curricula and syllabi about the connecting mathematics and engineering is actually practiced and assessed.

With such limited data at my disposal, obviously not much in the way of analysis has been conducted so far. Some initial observations can be made, however, as an indication of the types of findings the study will hopefully provide. For instance, comparing the syllabi for the first semester courses in civil engineering at the Norwegian university one notes certain differences between the mathematics course, Mathematics 1, and the engineering course, Technical design. In the mathematics course, which is given within six different engineering bachelor programs, there is no mention of engineering, or indeed anything outside of mathematics, except for the mention, in one of the learning goals, of "practical problems". On the contrary, in the Technical design course, the first learning goal states that the student should "understand the subject's correlation to the other courses in the civil engineering program". Also, the student should be able to "perform basic calculations on the properties of the main building materials." Hence, this course is seen as part of an overall civil engineering curriculum that makes use of mathematical techniques. The mathematics course, on the other hand, is presented as disjoint from the engineering context. Comparing this with the civil engineering bachelor at the Swedish university, the situation is largely similar, although in the aims of the first semester mathematics course, Basic course in mathematics, it is explicitly stated that the course is intended to “give basic mathematical knowledge and skills of importance for continued studies in mathematics and applied subjects”. In the content and the goals of the course, however, there is no mention of the world outside of mathematics, and thus mathematics is once again presented as something that is disjoint from the overall context, but that might be applied later. These initial observations are consistent with an applicationist understanding in the sense of Barquero, Bosch and Gascón (2011), although of course much more evidence needs to be gathered in order to be able to make such a claim with any force.

This project is still in its very early stages, and it is much too soon to be able to say anything of its potential impact. Still, it is hoped that the results of the project will give insight into the considerations needed to be made when designing engineering programs.

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